A model for the assembly map of bordism-invariant functors

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Given a functor $F : \mathcal{C} \to \mathcal{D}$ and a diagram X_α in $\mathcal{C}$, there is a map

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comparing the colimit of the $F(X_\alpha)$ and the image of the colimit under F. Of course, F preserves this colimit iff this map is an equivalence.

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Consider a group G, then for every $X \in \mathcal{C}$, we can consider the constant $\mathrm{B}G$-indexed diagram in $\mathcal{C}$ with value X. Its colimit is often written $\mathrm{B}G \otimes X$, and the map

$$\mathrm{B}G \otimes F(X) \longrightarrow F(\mathrm{B}G \otimes X)$$

is the *G-assembly map* of F at X.

In fact, we need not restrict to a group: if K is a space (in the higher categorical sense), there is again an assembly map

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In fact, the left hand side is uniquely described as the functor in K which preserves colimits and sends $*$ to $F(X)$. In particular, if the target of F is stable, then $K \mapsto K \otimes F(X)$ is the universal to approximate F by a homology theory.

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- (non-connective) algebraic K-theory $K(\mathcal{C})$, as a functor $\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$
- L-theory (quadratic, symmetric and many others as we will explain)

Proposition (Bass–Heller–Swan)

Let $\mathcal{C}$ be a stable category, then there is a natural splitting

$$K(S^1 \otimes \mathcal{C}) \simeq S^1 \otimes K(\mathcal{C}) \oplus N_+K(\mathcal{C}) \oplus N_-K(\mathcal{C})$$

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You might know this formula under a different formulation: for a spectrum X, $S^1 \otimes X \simeq X \oplus \Sigma X$ and if $\mathcal{C} = \mathrm{Perf}(R)$, then up to idempotent completion, $S^1 \otimes \mathrm{Perf}(R) \simeq \mathrm{Perf}(R[t, t^{-1}])$.

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Here, $S^1 \simeq \mathrm{B}\mathbb{Z}$, so the above fits in the following picture:

Conjecture (weak form of Farrell–Jones)

Let G be a torsionfree group, then the G-assembly map in K-theory is split-injective.

This is known for many G, but not in general.

Because there is a map $(-)^{\simeq} \to K$, it is straightforward to produce elements in the right hand side of the assembly map. The left hand side is harder to control, because it is not the K-theory of any stable category a priori.

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Any localizing invariant uniquely extends to $\mathrm{Pr}^{\mathrm{dual}}$, the category of dualizable objects in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ and strongly-continuous functors between them.

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In particular, if $\mathcal{C}$ is dualizable then so is $\mathrm{Shv}(X;\mathcal{C})$ (but it need not be compactly-generated, unless $\mathcal{C}$ is **and** X is a profinite space). And $\mathrm{Shv}(X;\mathcal{C})$ behaves like a *"2-categorical cohomology theory"*, so that for nice enough X:

$$K(\mathrm{Shv}(X;\mathcal{C})) \simeq K(\mathcal{C})^X$$

There is a way to formally dualise the construction of Shv:

Theorem (Bartels–Efimov–Nikolaus)

There is a map

$$\widehat{coShv}(X;\mathcal{C}) \longrightarrow X \otimes \mathcal{C} \simeq \mathrm{Fun}(X,\mathcal{C})$$

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However, in the battle, we lost our easy access to elements in K-groups: for a dualizable $\mathcal{C}$, the best there is a map $\mathcal{C}^{\omega,\simeq} \to K$ and there is little control on the compact objects of a dualizable category (it could very well be zero).

A Poincaré category $(\mathcal{C}, \Qoppa)$ is a stable category $\mathcal{C}$ with duality $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, i.e. D is an equivalence with D^{op} as its inverse, and an extra bit of datum, a linear functor $\Lambda : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ and a gluing functor

$$\Lambda(X) \longrightarrow \mathrm{map}_{\mathcal{C}}(X, D(X))^{\mathrm{tC_2}}$$

This is meant to encode which forms one is considering (e.g. quadratic, symmetric, skew-symmetric).

A Poincaré category $(\mathcal{C}, Ϙ)$ is a stable category $\mathcal{C}$ with duality $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, i.e. D is an equivalence with D^{op} as its inverse, and an extra bit of datum, a linear functor $\Lambda : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ and a gluing functor

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This is meant to encode which forms one is considering (e.g. quadratic, symmetric, skew-symmetric).

Note that pulling the gluing functor along the canonical map from homotopy fixed points yields:

$$\begin{array}{ccc} Ϙ(X) & \longrightarrow & B_{Ϙ}(X, X)^{\mathrm{hC_2}} \\ \downarrow & & \downarrow \\ \Lambda_{Ϙ}(X) & \longrightarrow & B_{Ϙ}(X, X)^{\mathrm{tC_2}} \end{array}$$

which knows all about the above datum: $\Lambda_{Ϙ}$ is the first Goodwillie derivative and $B_{Ϙ}$ is the second cross-effect. The duality is the curried functor (which a priori lands in $\mathrm{Ind}(\mathcal{C})$).

A Poincaré functor is an exact functor $f : \mathcal{C} \to \mathcal{D}$ equipped with a natural transformation $\eta : Ϙ \to \Psi \circ f^{\mathrm{op}}$, such that the induced map $f \circ D_{Ϙ} \implies D_{\Psi} \circ f$ is an equivalence.

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$$\mathsf{Hyp}(\mathcal{C}) \simeq (\mathcal{C} \oplus \mathcal{C}^{\mathrm{op}}, \mathsf{map}_{\mathcal{C}})$$

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There is a similarly defined notion of localizing invariants on $\mathbf{Cat}^{\mathrm{p}}$, the category of Poincaré categories and Poincaré functors.

Definition

A localizing invariant $E : \mathbf{Cat}^{\mathrm{p}} \to \mathcal{E}$ is *bordism-invariant* if it sends $\mathsf{Hyp}(\mathcal{C})$ to zero.

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The main example of such invariants is L-theory, $L : \mathbf{Cat}^{\mathrm{p}} \to \mathrm{Sp}$. This functor captures all of the variants of L-theory all at once: quadratic, symmetric, etc... by changing the Ω. It also allows them to talk to one another, for instance there is a map $(\mathcal{C}, \Omega^q_D) \to (\mathcal{C}, \Omega^s_D)$ which induces the quadratic to symmetric comparison on L-theory.

An *isotropic* subcategory $\mathcal{L}$ of $(\mathcal{C}, \Qoppa)$ is a coreflexive full subcategory such that $\Qoppa_{\mathcal{L}} \simeq 0$. In consequence, there is a fully-faithful $\mathrm{Hyp}(\mathcal{L}) \to (\mathcal{C}, \Qoppa)$.

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The isotropic subcategory $\mathcal{L}$ is a *Lagrangian* if $\mathcal{L}$, $D(\mathcal{L})$ jointly generate $\mathcal{C}$ as a stable category closed under retracts.

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Lemma

Let E be bordism-invariant and localizing. Suppose $(\mathcal{C}, \Omega)$ has a Lagrangian, then

$$E(\mathcal{C}, \Omega) \simeq 0$$

The category $\mathbf{Cat}^{\mathrm{p}}$ admits all limits and colimits, and they are preserved by the forgetful functor. In particular, given a Poincaré category $(\mathcal{C}, Ϙ)$, the right hand side of the assembly for a space X looks like

$$(X \otimes \mathcal{C}, X \otimes p_! Ϙ)$$

where $p : \mathcal{C} \to X \otimes \mathcal{C}$ is the canonical map of the colimit, $p_!$ is the left Kan extension functor and $X \otimes p_! Ϙ$ is the cotensor taken in the functor category.

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A localizing invariant is a *categorification* of the passage from $\mathcal{C}$ to $K_0(\mathcal{C})$. In particular, to produce the left hand side of the assembly, one strategy is to imitate the assembly map construction, except one categorical level higher, in similar fashion to the $\widehat{\mathrm{coShv}}(X; \mathcal{C})$.

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Note that if instead of a space K, we had started with a category I, the colimit of the constant $X \in \mathcal{C}$ is equivalent to $|I| \otimes X$ since the constant functor inverts all arrows and therefore factors through $I \to |I| \simeq I[\mathrm{all}^{-1}]$.

To categorify this idea, we can use that actually $\mathbf{Cat}^{\mathrm{Ex}}$ is a 2-category and admits something called *(op)lax colimits*. Roughly, oplax colimits are supposed to be objects with the same universal property as colimits, except with respect to *(op)lax cocones*, where all the triangles must now be given with a non-necessarily invertible transformation (the direction of which changes lax to oplax).

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The oplax colimit of a functor $F : I \to \mathbf{Cat}$ is its cocartesian unstraightening $\mathrm{Un}(F)$.

In general, even if F lands in $\mathbf{Cat}^{\mathrm{Ex}}$, this unstraightening is not stable. However, there is a universal way to map to a stable category while sending each exact sequence in the fibers $F(i)$ to exact sequence. We call this category $\mathrm{Un}^{\mathrm{Ex}}(F)$.

Given a functor $F : I \to \mathbf{Cat}^{\mathrm{p}}$, we might want to try to do the same thing. We expect that the oplax colimit will be a Poincaré category over the oplax colimit in stable categories. In particular, we have to produce

$$Ϙ : \mathsf{Un}^{\mathsf{Ex}}(F)^{\mathrm{op}} \to \mathsf{Sp}$$

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Since Sp is just a 1-category, any oplax cocone business must just be a regular cocone; in particular, we claim the following:

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Proposition (Levin–Nocera–S)

Given a functor $F : I \to \mathbf{Cat}^{\mathrm{p}}$, then the left Kan extensions of $\Omega_i : F(i)^{\mathrm{op}} \to \mathsf{Sp}$ to $\mathsf{Un}^{\mathsf{Ex}}(F)$ is functorial, and the resulting hermitian category:

$$\mathsf{Un}^{\mathsf{h}}(F) := (\mathsf{Un}^{\mathsf{Ex}}(F), \mathrm{colim}_i (p_i)_! \Omega_i)$$

is the oplax colimit in $\mathbf{Cat}^{\mathrm{h}}$, the (larger) category of hermitian category.

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Here, the 'hermitian' adjective is a weakening of the Poincaré, where the Ϙ is simply supposed to have a bilinear $B_Ϙ$ but no duality associated to it (and in particular, the functors are not duality preserving).

The reason the word hermitian appears is the following:

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The category $\mathrm{Un}^{\mathrm{h}}(F)$ need not be Poincaré, even if F is constant.

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For instance, for $I = \Delta^1$ and F constant equal to some $\mathcal{C}$, the Un^{h} is the hermitian structure $\Omega(X \to Y) := \Omega(Y)$, whose duality sends $X \to Y$ to $0 \to D(Y)$ and therefore is certainly not an equivalence ...

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There is half-a-hope however from the above:

Proposition (Levin–Nocera–S)

Suppose I is a strongly finite category ($=$ finite, enriched in finite spaces) then the duality on $\mathrm{Un}^{\mathrm{h}}(F)$ is always non-degenerate.

There is a situation however, where we are saved. Pick K a finite simplicial complex and consider $\mathrm{Face}(K)^{\mathrm{op}}$ the opposite of its poset of faces.

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Theorem (Levin–Nocera–S)

Let $F : \mathrm{Face}(K)^{\mathrm{op}} \to \mathbf{Cat}^{\mathrm{p}}$ be any functor. Then, $\mathrm{Un}^{\mathrm{h}}(F)$ is a Poincaré category.

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Why is there a miracle? Well the shape of the poset of faces of a simplicial complex is very self-dual: for instance, for Δ^1, it is given by

$$0 \leftarrow (0 \to 1) \to 1$$

A similar picture occurs for Δ^n, where the poset is given by $\mathrm{TwAr}(\Delta^n)$. This functor happens to preserve pushouts of simplicial complexes so this is all we needed to know.

In the previous situation, $\mathrm{Un}^{\mathsf{h}}(F)$ admits a rather explicit model: its underlying category is the category of sections of the cartesian fibration $\mathrm{Un}^{\mathsf{cart}}(F) \to \mathrm{Face}(K)^{\mathrm{op}}$ and the ♀ takes a section, viewed as a system $x_i \in F(i)$, and gives $\mathrm{colim}_i\, ♀(x_i)$, which is suitably functorial.

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Here it is particularly important to work with strongly finite categories. This is because sections of $\mathsf{Un}^{\mathsf{cart}}(F) \to \mathsf{Face}(K)^{\mathrm{op}}$ is actually the *oplax limit* of F, computed in **Cat** but also in $\mathbf{Cat}^{\mathrm{Ex}}$.

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In general, $\mathbf{Cat}^{\mathrm{Ex}}$ only has this kind of *lax semi-additivity* property for *finite* diagrams. This is in contrast with $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ in which for instance, colimits indexed by spaces coincide with limits indexed by the same space, and more generally, oplax colimits and oplax limits indexed by a category coincide. The *strong-finiteness* is a sufficient condition for this behaviour to descend to $\mathbf{Cat}^{\mathrm{Ex}}$.

Now, don't get ahead of yourselves:

Warning

Even in the previous situation, the maps $F(i) \to \mathrm{Un}^{\mathrm{h}}(F)$ need not be duality preserving. In particular, $\mathrm{Un}^{\mathrm{h}}(F)$ is not the oplax colimit in $\mathbf{Cat}^{\mathrm{p}}$ since its oplax cocone does not even belong there.

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In general, only the maps corresponding to inclusions of 0-simplicies are duality-preserving; in fact, the inclusion of a k-face actually shifts the duality by k.

Given a finite simplicial set K, note that the associated space $|K|$ is also $|\operatorname{Face}(K)^{\mathrm{op}}|$. Because oplax colimits are functorial, givem $F : |K| \to \mathbf{Cat}^{\mathrm{p}}$ there is a functor

$$\operatorname*{oplaxcolim}_{\operatorname{Face}(K)^{\mathrm{op}}} F \longrightarrow \operatorname*{oplaxcolim}_{|K|} F \simeq \operatorname{colim}_{|K|} F$$

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The conclusion comes from the fact that Poincaré categories are closed under colimits in hermitian ones.

The second part is rather formal: because $\mathrm{Face}(K)^{\mathrm{op}} \to |K|$ is a localisation, it must be that the induced functor on oplax colimits also is (roughly because for a functor to descend along the oplax colimit, it suffices that it inverts the relevant maps in the diagram).

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The first part of the above proposition is however more technical. The core strategy already implements something we have talked about: to prove that the duality is an equivalence is a local condition. Indeed, it asks whether the map

$$X \longrightarrow DD(X)$$

is an equivalence. In particular, it suffices that every X is in the image of a duality preserving.

But every simplicial complex is glued from Δ^n, and this makes oplax colimits into a *2-categorical homology theory*, in the following sense.

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Proposition (Levin–Nocera–S)

If $L' = L \coprod_K \Delta^n$ is a pushout of finite simplicial complexes and $F : \mathrm{Face}(L')^{\mathrm{op}} \to \mathbf{Cat}^{\mathrm{p}}$, then the square

$$\begin{tikzcd} \operatorname{oplaxcolim}_{\mathrm{Face}(K)^{\mathrm{op}}} F \arrow[r] \arrow[d] & \operatorname{oplaxcolim}_{\mathrm{Face}(\Delta^n)^{\mathrm{op}}} F \arrow[d] \\ \operatorname{oplaxcolim}_{\mathrm{Face}(L)^{\mathrm{op}}} F \arrow[r] & \operatorname{oplaxcolim}_{\mathrm{Face}(L')^{\mathrm{op}}} F \end{tikzcd}$$

is a pushout square, whose vertical legs are kernel inclusions.

In particular, this also implies that the kernels maps to one another and in fact, can be exhausted from the inclusions of Δ^n. Hence, we are reduced to

Lemma

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which is a (technical) computation, but doable nonetheless.

In fact, being careful about the computation, we find more:

Proposition

The kernel in the case $K = \Delta^n$ admits a Lagrangian.

This last fact has all sorts of cool consequences for bordism-invariants functors. It means that for $K = \Delta^n$, the map

$$E(\operatorname*{oplaxcolim}_{\mathsf{Face}(\Delta^n)^{\mathrm{op}}} F) \longrightarrow E(\operatorname{colim}_{|\Delta^n|} F) \simeq F(*)$$

is an equivalence, hence the left hand side models the assembly (for a contractible space).

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$$E(\phi_K) : E(\underset{\mathrm{Face}(K)^{\mathrm{op}}}{\mathrm{oplaxcolim}}\, F) \longrightarrow E(\mathrm{colim}_{|K|}\, F)$$

is a model of the assembly for all finite simplicial complexes K. In particular, the left hand side does not depend on the specific simplicial complex realizing to $|K|$.

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This holds more generally if E preserves filtered colimits.

Theorem (Levin–Nocera–S)

For every bordism-invariant localizing $E : \mathbf{Cat}^{\mathrm{p}} \to \mathcal{E}$ and every finite simplicial complex K, the map

$$E(\phi_K) : E(\operatorname*{oplaxcolim}_{\mathrm{Face}(K)^{\mathrm{op}}} F) \longrightarrow E(\mathrm{colim}_{|K|} F)$$

canonically identifies with the assembly map. Moreover, its kernel is $E(\ker \phi_K)$ because ϕ_K was the cofiber of its fiber.

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Note that having a Lagrangian is not a property stable under pushout (it is only stable under *Lagrangian-preserving* functors), so we have not proved that L-theory commutes with arbitrary colimits of spaces (which is false, anyway).

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In some easy cases (like S^1), the Lagrangians do glue, and so we get a L-theoretic Bass–Heller–Swan (with even the added benefit that there are no nil-terms, and we can even twist by duality-preserving automorphisms):

$$S^1 \otimes L^{\langle -\infty \rangle}(\mathcal{C}, \Omega) \simeq L^{\langle -\infty \rangle}(S^1 \otimes (\mathcal{C}, \Omega))$$

A few bullets points about the future:

- Is there a way to do this for non-necessarily bordism-invariant functors, without having to do *dualizable Poincaré categories*?
- Is there a way to compute more, non-trivial examples of kernels vanishing?
- Is there a way to deduce some inheritence properties of Farrell–Jones type statements?