Topological Hochschild homology, traces and higher categories

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Abstract

This is the typed notes of a lecture given in Bielefeld in Winter 2025/2026, as a follow-up to the course Higher categories and algebraic K-theory III taught by Fabian Hebestreit. Their goal is roughly to explain a modern point of view on THH and slowly build towards to the relationship with K-theory through trace methods.

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The main character of these notes is THH, topological Hochschild homology, a spectrum which can be associated to a ring, a ring spectrum, a stable category with even suitable coefficients. THH is a very rich object: it enjoys an interesting functoriality, has plenty of mysterious extra structure, is linked with many differents other invariants of interests and comes with a famed history. As our point of view will be quite modern, we want to recall a few elements of history at the very beginning.

Before topological Hochschild homology, there was simply Hochschild homology, which we will denote $\mathrm{HH}_{\mathbb{Z}}$. Given a ring R and a R-bimodule M, $\mathrm{HH}_{\mathbb{Z}}(R, M)$ is traditionally as the Tor-groups of R and M viewed as modules over $R \otimes R^{\mathrm{op}}$; our higher categorical point of view allows to simply write

$$\mathrm{HH}_{\mathbb{Z}}(R, M) := R \otimes^{\mathrm{L}}_{\mathbb{Z}} M$$

where for the first and last time, we added a superscript L to insist on the fact that this tensor is derived, that is, taken in the presentable stable category $D(\mathbb{Z})$ of derived $\mathbb{Z}$-modules which we will simply denote $\mathrm{Mod}(\mathbb{Z})$, keeping in with modern fashion.

The name comes from Hochschild, who introduced it in a paper as the homology of an explicit complex (the so-called Hochschild complex). In modern terms, it is obtained via the Dold-Kan correspondence from the following simplicial object:

$$M \otimes R^{\otimes n} \rightrightarrows \dots \rightrightarrows M \otimes R \rightrightarrows M$$

using the R-linear multiplication map on M on the k^{th}-component. Note in particular that $\pi_0 \mathrm{HH}_{\mathbb{Z}}(R, M) \simeq R/[R, M]$, and the canonical map $\mathrm{tr} : \mathrm{Proj}(R)^{\simeq} \to R/[R, R]$ factors through $K_0(R)$, because it is in particular additive. In fact, the trace map lifts to the whole spectrum, to something called the *Dennis trace map*:

$$K(R) \longrightarrow \mathrm{HH}_{\mathbb{Z}}(R, R)$$

Even for non-K-theorists, Hochschild homology is quite an interesting object. For a smooth commutative algebra A over a field k of characteristic zero, $\pi_n \mathrm{HH}_{\mathbb{Z}}(A/k, A/k)$ coincides with the Kähler differentials $\Omega^n_{A/k}$, and for general non-smooth A, receives at least a comparison map.

It also comes with an action of the circle S^1, related to the de Rham differentials $\Omega^n_{A/k} \to \Omega^{n+1}_{A/k}$ (we refer to Matthew Mororw's notes for more details in this direction). This S^1-action only exist when the bimodule is the ring itself, and can be understood from the perspective of cyclic objects, as the geometric realization of cyclic object carries canonically such an action.

Taking homotopy fixed points and the Tate cohomology (in the $\mathbb{Z}$-linear world) for the S^1-action yields spectra called *(negative) cyclic homology* HC^- and HP. There is a map $\mathrm{HC}^- \to \mathrm{HP}$ whose fiber we call HC, the *cyclic homology* — traditionally, we should incorporate a shift because the norm map for S^1 has a shift (see Corollary I.4.3 of [NS17]):

$$\Sigma(-)_{\mathrm{h}S^1} \xrightarrow{\mathrm{Nm}_{S^1}} (-)^{\mathrm{h}S^1} \longrightarrow (-)^{\mathrm{t}S^1}$$

so that HC coincides with homotopy orbits for the S^1-action but this introduce an annoying shift in the notation everywhere else. These invariants were discovered first by Connes and Tsygan, without quite realizing the S^1-action at first, and it is under Connes' impulse that the cyclic category was introduced to formalize this action and reinterpret the earlier construction in this framework.

The trace map for $\mathrm{HH}_{\mathbb{Z}}$ is not very interesting because Hochschild homology is often far too simple to tell interesting things in K-theory: for instance there is no higher Hochschild homology for $\mathbb{Z}$ so this map loses the information on higher K-groups of $\mathbb{Z}$, and similarly for $\mathbb{F}_p$, $\mathrm{K}(F_p)$ is in odd degrees and $\mathrm{HH}_{\mathbb{Z}}(\mathbb{F}_p)$ is even. However, this trace map is S^1-equivariant for the trivial action on K-theory. In particular, it lifts to a map $\mathrm{K} \to \mathrm{HC}^-$ and rationally, it also vanishes on HP hence lifts to $\mathrm{K} \otimes \mathbb{Q} \to \mathrm{HC} \otimes \mathbb{Q}$. This refined *cyclic trace map* is actually able to capture more on K-theory. A result of Goodwillie [Goo86] states that if $R \to S$ is surjective with nilpotent kernel, then

$$\mathrm{fib}(\mathrm{K}(R) \to \mathrm{K}(S)) \longrightarrow \mathrm{fib}(\mathrm{HC}(R) \to \mathrm{HC}(S))$$

is a rational equivalence (i.e. an equivalence after tensoring with $\mathbb{Q}$). This is quite a striking result, as computations in K-theory are really hard, whereas HC is a manageable object to compute. Unfortunately, it just breaks down away from characteristic zero. This is where THH enters the story.

The insight, due to Goodwillie and Waldhausen, is that K-theory, unlike HH, is not really a "linear" object, e.g. $\mathrm{K}(\mathbb{F}_p)$ is not a $\mathbb{F}_p$-module. It mostly lives over the initial (non-zero) ring ... but this is *not(!)* $\mathbb{Z}$ in homotopy theory, but the sphere spectrum $\mathbb{S}$. They wondered if there was a "topological" refinement of HH (in the sense that it understood more than just $\pi_0\mathbb{S} \simeq \mathbb{Z}$ but the topology above) and this replacement should make the statements hold integrally.

In fact, it was known that *stable K-theory*, the invariant obtained from K-theory by forcefully adding a dependence in the bimodule variable via the square-zero extension $\mathrm{K}(R \oplus M)$ and then forcing it to be M-linear, was rationally Hochschild homology and it was expected that the integral object was this topological Hochschild homology, a conjecture that made it into Goodwillie's 1990 ICM address.

Of course, this predates more higher categorical technology so it took Bökstedt some amount of effort to define properly THH, and study the extra structure — with Hsiang and Madsen in [BHM93], they described that not only did $\mathrm{THH}(R)$ have a S^1-action, it also had a cyclotomic structure which in modern terms we would describe as S^1-equivariant maps

$$\phi_p : \mathrm{THH}(R) \longrightarrow \mathrm{THH}(R)^{\mathrm{tC}_p}$$

Using this structure, one can form topological version of the periodic and negative theories we introduced earlier, namely we let $\mathrm{TC}^-(R) := \mathrm{THH}(R)^{\mathrm{h}S^1}$, $\mathrm{TP}(R) := \mathrm{THH}(R)^{\mathrm{t}S^1}$, but the correct replacement of HC actually involves the cyclotomic structure. In formula, following the Nikolaus–Scholze approach of [NS17], one lets:

$$\mathrm{TC}(R) := \mathrm{Eq}\left(\mathrm{THH}(R)^{\mathrm{h}S^1} \underset{(\phi_p^{\mathrm{h}S^1})}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_{p \text{ prime}} (\mathrm{THH}(R)^{\mathrm{tC}_p})^{\mathrm{h}S^1} \right)$$

The resulting invariant is called *topological cyclic homology.* About at the same time, Dundas–McCarthy proved in [DM94] that THH was indeed stable K-theory for connective rings and connective bimodules, and after some more efforts, they produced an integral version of Goodwillie's theorem, namely that if $R \to S$ is map of connective ring spectra such that on π_0, it is surjective with nilpotent kernel, then

$$\mathrm{fib}(\mathrm{K}(R) \to \mathrm{K}(S)) \xrightarrow{\simeq} \mathrm{fib}(\mathrm{TC}(R) \to \mathrm{TC}(S))$$

is an equivalence. The proof of this result is quite technical, and relies on both various simplicial comparisons and the calculus of functors of Goodwillie — something that Goodwillie had already envisioned for his result in [Goo86] even if he did not use it in the end. This result is particularly key to compute K-theory of more complicated rings, like $\mathbb{Z}/p^n\mathbb{Z}$ when $n \geq 2$, see [AKN24].

The goal of this course, or what we want to achieve at the end of multiple courses following one another, is to *explain* this result, and the word has been italicized because we do not simply want to give a presentation of the proof with minor modern improvement but truly a different treatment of it, which follows the ideas of the series of papers [HNS24, HNRS25a, HNRS25b] — which are also currently not all been publicly released.

There are two major differences we want to implement: the first is to move away from rings, or even ring spectra and try to understand this story at the level of stable categories, sometimes idempotent-complete or even large dualizable following insights of [Efi24]. In K-theory, this has always been somewhat standard ever since Quillen's seminal work on higher algebraic K-theory [Qui73] but references for THH of stable categories are few and far between. We claim that done correctly, this will allow, just as in K-theory, to turn THH from an object realized by a certain construction and the structure therein inherited from special properties of this construction, into an object having a universal property and us being able to prove central features of THH via the study of the often simpler property.

In THH, unlike in K-theory, it is central to implement this *with coefficients*. These coefficients, which generalize bimodules over a ring, are bi-exact functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$. We will also explain why these are naturally the "coefficients of a linear theory" over $\mathbf{Cat}^{\mathrm{Ex}}$ — by this we mean functors $F(\mathcal{C}, -)$ where the blank variable is colimit-preserving or at least exact — by identifying them as $\mathcal{C}$ varies with the category $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, the *tangent bundle of* $\mathbf{Cat}^{\mathrm{Ex}}$ which is the abstract category of coefficients of linear theories over $\mathbf{Cat}^{\mathrm{Ex}}$.

In this world, we will furnish a universal property for THH, which will use that $\mathrm{THH}(\mathcal{C}, M)$ is linear in the M-variable and the other key feature of THH we have not mentioned: its invariance under cyclic permutations. More precisely if M is a $(\mathcal{C}, \mathcal{D})$-bimodule and N a $(\mathcal{D}, \mathcal{C})$-bimodule, then there is an equivalence:

$$\mathrm{THH}(\mathcal{C}, M \otimes_{\mathcal{D}} N) \simeq \mathrm{THH}(\mathcal{C}, N \otimes_{\mathcal{D}} M)$$

This cyclic invariance is one of the defining feature of the trace. The reader fond of linear algebra might know for instance that a linear form $f : M_n(\mathbb{R}) \to \mathbb{R}$ which has the cyclic invariance is necessarily a multiple of the trace, namely $f(-) = f(E_1)\,\mathrm{tr}(-)$.

The notion of trace is one that can be defined extremely generally. We will show that THH *is* a trace, in the category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ of (large) presentable stable categories, which will take us into a expository panorama of large categories in the higher world. In fact, following Ramzi in his thesis, we will show that the uniqueness characterization of the trace *lifts* to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, namely the functor

$$\mathrm{ev}_{\mathbb{1}} : \mathrm{Fun}^{\mathrm{cyc},\mathrm{fbw-L}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \longrightarrow \mathcal{E}$$

which evaluates at the unit of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ a cyclic-invariant, colimit-preserving in the coefficients, functor to a presentable stable $\mathcal{E}$, is an equivalence with inverse $X \mapsto X \otimes \mathrm{THH}$.

In fact, there is a refinement of this story that is central to trace methods in K-theory. Let us first recall some linear algebra: given a real matrix M over $\mathbb{R}$, one can compute the whole Taylor tower of $\det(I + tM)$. In fact, it is even easier to express after passage to the logarithm:

$$\ln\det(I + tM) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}\,\mathrm{tr}(M^n)t^n$$

We claimed that THH was a refinement of the trace and our refinement of $\ln\det(I + tM)$ is the fiber $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, M) := \mathrm{fib}(\mathrm{K}(\mathcal{C} \oplus M) \to \mathrm{K}(\mathcal{C}))$ where $\mathcal{C} \oplus M$ is a categorical version of the square-zero extension, details of which we won't go into now. Note that as the name suggest, $\mathrm{K}^{\mathrm{cyc}}$ has the cyclic invariance of the trace (in fact, this is also true of $\det(I+M)$ and is known under the name of *Weinstein-Aronszajn identity*). In particular, forcefully imposing cyclic K-theory to commute with colimits in the M variable will give a point in $\mathrm{Fun}^{\mathrm{cyc,fbw-L}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$, i.e. this derivative is of the form $X \otimes \mathrm{THH}$ and $X \simeq \mathbb{S}$ by a previously mentioned result of Dundas-McCarthy.

More generally, one can show that a n-excisive, finitary, additive, cyclic invariant $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ promotes to n-truncated polygonic objects in $\mathcal{E}$, i.e. that one can record functorially the data $F(\mathcal{C}, M^{\otimes k})$ for $1 \leq k \leq n$ and they are related by maps

$$\phi_{k,l} : F(\mathcal{C}, M^{\otimes k}) \longrightarrow F(\mathcal{C}, M^{\otimes kl})^{\tau \mathrm{C}_l}$$

which are C_k-invariant and exist for $kl \leq n$. The target is the *proper* Tate construction, i.e. the Tate construction with respect to the family of proper subgroups of C_l instead of the usual (which is with respect to the trivial family of subgroups). This provides a functor

$$\mathrm{Fun}^{\mathrm{cyc,fbw}-n\mathrm{exc,add},\omega}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \longrightarrow \mathrm{Pgc}_{\leq n}(\mathcal{E})$$

which evaluates at the unit of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ the refined functor valued in polygonic objects. The extra hypotheses (finitary, additive) are precisely added so that this functor is still an equivalence. The inverse is given in formula by $X \mapsto \mathrm{TR}_n(X \otimes \mathrm{THH}(-))$ where the tensor product is using that $\mathrm{Pgc}_{\leq n}(\mathcal{E})$ is tensored over $\mathrm{Pgc}_{\leq n}(\mathrm{Sp})$, that THH admits such a structure and $\mathrm{TR}_n : \mathrm{Pgc}_{\leq n}(\mathcal{E}) \to \mathcal{E}$ is the right adjoint of the trivial functor.

In particular, TR_n is trying to glue back the extra data supplied by the polygonic spectra, in a way not too dissimilar to truncating the sum of traces in the Taylor tower of $\ln\det(I+tM)$. In fact, taking the n-excisive approximation of cyclic K-theory gives a functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ with all of the extra properties which coincides with $\mathrm{TR}_n(\mathrm{THH})$, i.e. up to some extension problems which mean we cannot write a direct sum, the formula for $\ln\det(I+tM)$ holds also in the world of stable categories with coefficients.

In fact, and at least for the case of square-zero extensions, the Dundas-Goodwillie-McCarthy theorem can be understood as a phenomenon of both cyclic K-theory and cyclic TC converging to the limit of their Taylor tower, which also happen to coincide.

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1 How to tame your large category

Large categories can be scary. The goal of this section is to give the reader some tools to turn their large, smelly, hirsute category into a well-behaved, groomed and all-together *presentable* category.

1.1 The amazing category ~~of spaces of homotopy-types of anima of groupoids~~ $\mathcal{S}$

Let us begin by a example, the poster child of a nice, large category: the category $\mathcal{S}$ of what we will call spaces or groupoids but feel free to use any other word you prefer, like anima or homotopy-type. The category $\mathcal{S}$ is the full subcategory of $\mathbf{Cat}$ spanned by those categories where every arrow is invertible.

Proposition 1.1.1 The inclusion $\mathcal{S} \to \mathbf{Cat}$ admits both a left and a right adjoint. The former is denoted $|\cdot|$, and computed by forming the localization at all arrows and the latter, denoted $(-)^{\simeq}$, is obtained as the wide subcategory spanned by invertible arrows.

In particular, the above provides a rather robust way of computing colimits and limits in $\mathcal{S}$, since one can use the machinery developed for categories. We recall the following statement, which is paramount to compute colimits of categories and was proven in Fabian's earlier lecture.

Lemma 1.1.2 Let $F : \mathcal{C} \to \mathbf{Cat}$ and denote $\mathrm{Un}(F) \to \mathcal{C}$ the cocartesian unstraightening of F. Then, $\operatorname{colim} F$ is the localisation of $\mathrm{Un}(F)$ at the cocartesian edges.

As a sanity check, remark that if F is space-valued, then every arrow in $\mathrm{Un}(F)$ factors as a cocartesian edge followed by an equivalence, so $\operatorname{colim} F$ is indeed a space.

To compute limits in $\mathbf{Cat}$, one must take the category of sections of the unstraightening of a functor instead, and then take the full subcategory spanned by those γ such that for every $\alpha : i \to j$, the induced map $F(\alpha)(\gamma(i)) \to \gamma(j)$ is an equivalence (or the other way around, depending of whether one takes the cocartesian or the cartesian unstraightening). The reader who is not familiar with these ideas is invited to try to compute for instance pullbacks of categories this way, as we will often use the concreteness of this construction.

Corollary 1.1.3 The category $\mathcal{S}$ is complete and cocomplete.

Let us further analyse colimits in $\mathcal{S}$:

Proposition 1.1.4 Every object in $\mathcal{S}$ is a colimit of $*$. In fact, $\mathcal{S}$ is *freely* generated under colimits, in the sense that the inclusion $i : \{*\} \to \mathcal{S}$ induces an equivalence

$$i^* : \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$$

for every cocomplete $\mathcal{C}$, where $\mathrm{Fun}^{\mathrm{L}}$ denotes the full subcategory of colimit-preserving functors.

Proof. Using Lemma 1.1.2, the first claim is immediate as any $X \in \mathcal{S}$ is also the unstraightening of the associated functor $\mathrm{cst}(*) : X \to \mathcal{S}$ of the constant functor equal to $*$.

Let us also include a more "topological" explanation. Recall that up to weak equivalence, every (nice-enough) topological space is a CW-complex, with possibly infinitely many cells in each dimension. In particular, each cell is built out of spheres $S^n := \Sigma^n S^0$ where $S^0 := \{*\} \coprod \{*\}$ and disks which are contractible i.e. homotopic to a point. Since the gluing in CW-complexes happens along cofibrations, any presentation of a CW-complex gives rises to a colimit-presentation of the associated homotopy type.

We now prove that i is an equivalence. First recall from say [Lur08, Proposition 4.3.3.7] that

$$i^* : \mathrm{Fun}(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$$

has a left adjoint $i_!$ which is fully-faithful and given by the left Kan extension functor. Unraveling the formula for left Kan extension, for a point $A : * \to \mathcal{C}$, $i_! A$ is the functor described pointwise by

$$X \mapsto \operatorname*{colim}_{p : * \to X} A(*)$$

which we will often write $X \otimes A(*)$. This colimit does indeed exist since $\mathcal{C}$ is cocomplete and $i_!$ is fully-faithful because i is (see §4.3.2 of [Lur08]). Because colimits commute with other colimit or more generally, thanks to the lemma below, this adjoint lands in the full subcategory $\mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{C})$ and therefore the whole adjunction descends.

In particular, as the adjoint of a fully-faithful functor, i^* is a localisation, namely at the collection of arrow $\mathcal{W} := \{i_! i^*(F) \longrightarrow F\}$ — by this, we mean that any functor Φ which inverts those arrows must factor essentially uniquely through i^*: this is obvious since the collection of arrows gives the factorization $(\Phi \circ i_!) \circ i^*$. Now notice that i^* is conservative: if $F \to G$ is a natural transformation of colimit-preserving functors $\mathcal{S} \to \mathcal{C}$ such that $F(*) \to G(*)$ is an equivalence, we claim that $F(X) \to G(X)$ is also always an equivalence. This follows easily from choosing a presentation of X as a colimit of $*$ which exists by the first part.

To conclude, we remark that a conservative localisation is necessarily a localisation at no non-trivial arrows (or directly that a functor which is conservative and has a fully-faithful adjoint is an equivalence from the triangle identities), hence an equivalence. □

If $\mathcal{C}$ is a category, we write $\mathcal{P}(\mathcal{C})$ for the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of presheaves and $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ for the Yoneda embedding. We used the following Lemma in the above proof: The order is fucked

Lemma 1.1.5 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with target a cocomplete category, and let $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ be the Yoneda embedding. Then, the functor $j_!F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ preserves colimits.

Proof. We follow roughly Theorem 8.4.3.5 in [Lur18, Tag 03WH]. Since we can test along mapping spaces (i.e. it suffices to show that $\mathrm{Map}(j_!F(-), X)$ sends colimits to limits), we reduce without loss of generality to the case where $\mathcal{D} = \mathcal{S}^{\mathrm{op}}$.

Note that $(j_!)^{\mathrm{op}}$ identifies with the functor which right Kan extend along j^{op}, which is just the Yoneda embedding of $\mathcal{C}^{\mathrm{op}}$. Under this identification, we have to justify that this functor restricts to

$$\mathrm{Fun}(\mathcal{C}, \mathcal{S}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \longrightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}), \mathcal{S}^{\mathrm{op}})$$

i.e. that if $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a presheaf, then j_*F sends colimits to limits. But by the Yoneda lemma, there is an equivalence

$$j_*F \simeq \mathrm{Nat}(-, F)$$

since both sides have the same universal property.

A different proof of this claim is to remark that $j_!F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ has a right adjoint given by

$$R_F : X \in \mathcal{D} \longmapsto \mathrm{Map}_{\mathcal{D}}(F(-), X) \in \mathcal{P}(\mathcal{C})$$

which can be checked by the local criterion for adjunctions. □

In fact, the proof of Proposition 1.1.4 generalizes to the following:

Proposition 1.1.6 Let $\mathcal{D}$ be a cocomplete category and $\mathcal{C}$ a small category, then, restriction along the Yoneda lemma induces an equivalence:

$$j^* : \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

We say that $\mathcal{P}(\mathcal{C})$ is *freely generated by* $\mathcal{C}$ *under colimits.*

Proof. Since $\mathcal{D}$ has small colimits, j^* has a left adjoint $j_!$ given by left Kan extension along j, which indeed lands in colimit-preserving functor and is fully-faithful since the Yoneda lemma guarantees that j is fully-faithful. Hence, as in Proposition 1.1.4, j^* is a localisation and it suffices to check that it is conservative and by similar arguments, this reduces to the fact that every $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a colimit of representable functors.

Indeed, there is a map of spaces, natural in X, which we can obtain by the evaluation of the counit of the above adjunction for the functor $\mathrm{id} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$,

$$\operatorname*{colim}_{j(Y)\to\phi} \mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \phi(X)$$

We claim this map is an equivalence. Note that this colimit is indexed by $\mathcal{P}(\mathcal{C})_{/\phi} \times_{\mathcal{P}(\mathcal{C})} \mathcal{C}$ which by the Yoneda lemma, corresponds to the cartesian unstraightening of the functor $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. But the functor $\mathrm{Map}(X, p(-)) : \mathrm{Un}^{cart}(\phi) \to \mathcal{S}$ factors through the projection $p : \mathrm{Un}^{cart}(\phi) \to \mathcal{C}$.

We can use Lemma 1.1.2 to compute this colimit. The cocartesian unstraightening of $\mathrm{Map}(X, p(-))$ is given by $\mathcal{C}_{X/} \times_{\mathcal{C}} \mathrm{Un}^{cart}(\phi)$; this category receives a map from $\phi(X)$ thanks to the commutative diagram:

$$\begin{array}{ccc} \phi(X) & \xrightarrow{\mathrm{cst}(\mathrm{id}_X)} & \mathcal{C}_{X/} \\ \downarrow{\scriptstyle\subset} & & \downarrow \\ \mathrm{Un}^{cart}(\phi) & \longrightarrow & \mathcal{C} \end{array}$$

Now, $\phi(X)$ is a space so it suffices to argue that $\phi(X) \to \mathcal{C}_{X/} \times_{\mathcal{C}} \mathrm{Un}^{cart}(\phi)$ is a weak homotopy equivalence. This follows from the fact that this functor has an adjoint, which we can describe as sending $(f : X \to Y, y \in \phi(Y))$ to the $\phi(f)(y) \in \phi(X)$ — this defines a left adjoint because $(\mathrm{id}_X : X \to X, x \in \phi(X))$ is initial in each slice. This concludes. □

In particular in the proof, we obtained:

Corollary 1.1.7 Every presheaf is a colimit of representable presheaves, i.e. the image of j generates $\mathcal{P}(\mathcal{C})$ under colimits.

We now turn to:

Definition 1.1.8 A *finite space* is an object $X \in \mathcal{S}$ which can be obtained as a finite colimit of $*$, i.e. in the smallest full subcategory of $\mathcal{S}$ closed under coproducts and pushouts and containing the point.

Note that it is hard to not be self-referential in defining the finiteness notion. The above is not but it was implemented the fact that iterated coproducts and pushouts produce all finite diagrams, which one might want as a property and not a definition.

An equivalent definition is that a category $\mathcal{C}$ is finite if and only if there exists a simplicial set weakly-equivalent to $\mathcal{C}$ with finitely many non-degenerate simplices. It holds that the category of finite categories is the smallest closed under pushouts and coproducts and containing both $*$ and $\{0 \to 1\}$. Note that since $\mathcal{S} \to \mathbf{Cat}$ preserves colimits, finite spaces are also legitimate finite categories and they span the further subcategory which is only generated by $*$; in particular, finite spaces are therefore equivalently those who can be modelled by Kan complexes with finitely many non-degenerate simplices and those obtained by finitely many pushouts and coproducts out of $*$.

We now want to explain what ones needs to do to recover the whole category $\mathcal{S}$ from its finite objects.

Definition 1.1.9 A diagram category I is called *filtered* if for every finite category $\mathcal{C}$, the diagonal functor $\mathrm{cst} : I \to \mathrm{Fun}(\mathcal{C}, I)$ sending i to the constant functor $\mathcal{C} \to I$ with value i is cofinal.

Differently stated thanks to Quillen's Theorem A [Lur08, Theorem 4.1.3.1], $\mathrm{cst} : I \to \mathrm{Fun}(\mathcal{C}, I)$ is cofinal if for every $f : \mathcal{C} \to I$, the category of diagrams $\mathrm{Fun}(\mathcal{C}, I)_{f/} \times_{\mathrm{Fun}(\mathcal{C},I)} I$, whose objects are natural transformations $\{f \longrightarrow \mathrm{cst}(i)\}$ and maps are induced by maps $i \to j$ in I making the associated diagrams commute, is weakly contractible.

■ **Example 1.1.10** Right adjoint functors are always cofinal by virtue of the categories required to be weakly contractible having an initial object, hence categories with finite colimits are filtered. ■

Remark 1.1.11 Actually, it suffices that each $\mathrm{Fun}(\mathcal{C}, I)_{f/} \times_{\mathrm{Fun}(\mathcal{C},I)} I$ is non-empty for it to be weakly contractible (see Proposition 9.1.1.18/Tag 02PJ of [Lur18]).

In particular, since cofinal maps are weak equivalences, I is non-empty using the case $\mathcal{C} = \emptyset$. Moreover, using the case of finite sets, it follows that one can find a cone point for every pair of objects of I as well as an equalizing morphism for any two pair of morphisms. In particular, filtered 1-categories are filtered in the higher sense as well.

Writing a space as the filtered colimit of a finite skeleta, we get:

Corollary 1.1.12 Every object $X \in \mathcal{S}$ is a filtered colimit of finite spaces.

More is actually true, as we will soon show, but first let us introduce another notion, which we will quickly relate to finiteness:

Definition 1.1.13 A space X is *compact* if the functor $\mathrm{Map}(X, -) : \mathcal{S} \to \mathcal{S}$ commutes with filtered colimits.

■ **Example 1.1.14** The empty space $\emptyset$ is compact. The point $*$ is compact. ■

In fact, it is possible to recognize filtered categories by how the colimit functor valued in $\mathcal{S}$ behaves:

Proposition 1.1.15 In $\mathcal{S}$, filtered colimits commute with finite limits. In particular, finite spaces are compact.

Proof. Let us explain quickly the second part: as a functor in X, $\mathrm{Map}(X, -)$ sends finite colimits to finite limits. In particular, a finite colimit of compact objects stays compact (note that this actually holds for any category $\mathcal{C}$, as it only uses the commutation at the target). This concludes using the previous example.

For any finite diagram I, any filtered J and any functor $X : I \times J \to \mathcal{S}$, there is a natural map

$$\eta : \operatorname*{colim}_{j\in J} \lim_{i \in I} X(i,j) \longrightarrow \lim_{i\in I} \operatorname*{colim}_{j\in J} X(i,j)$$

and we have to check it is an equivalence. Note that we can reduce to the case where the finite limit is a pullback or terminal. The latter case is straightforward by virtue of filtered categories being contractible.

We will not give a proof for the case of pullbacks, but let us sketch a strategy. Because we have access to particularly explicit descriptions in the case of sets, it is easier to check that this property holds there. One strategy, which is the one of [Lur08], is therefore to push this fact through to nice topological spaces and then through the localisation, see for instance [Lur08, Proposition 5.3.3.3]. Another way of presenting this idea is through [Lur18, Tag 05XW], which is less model-dependant. ☐

This makes $\mathcal{S}$ compactly-generated, i.e. every space is a filtered colimit of compact spaces; we will return to this property later. In a different direction, let us also say that this commutation property of filtered colimits is an equivalent characterization of filtered diagrams:

Corollary 1.1.16 A category J is filtered if and only if the functor $\operatorname{colim}_J : \operatorname{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves finite limits.

Proof. Given the above, we are reduced to prove that if colim_J preserves finite limits, then the functor $\operatorname{cst}_* : J \to \operatorname{Fun}(X, J)$ is cofinal for every finite category K. As we explained earlier, it suffices to check that for every $F : K \to J$, the category $\operatorname{Fun}(K,J)_{F/} \times_{\operatorname{Fun}(K,J)} J$ is weakly contractible. Another description of this category is the unstraightening of the functor $j \in \mathcal{J} \mapsto \operatorname{Nat}(F, \operatorname{cst}(i)) \simeq \lim_{k\in K^{\mathrm{op}}} \operatorname{Map}(F(k), j)$.

Now, since K^{op} is again finite, we know that

$$\eta : \operatorname*{colim}_{j\in J} \lim_{k\in K^{\mathrm{op}}} \operatorname{Map}(F(k), j) \longrightarrow \lim_{k\in K^{\mathrm{op}}} \operatorname*{colim}_{j\in J} \operatorname{Map}(F(k), j)$$

is an equivalence. In particular, the left hand side is also the localisation of $\operatorname{Fun}(K,J)_{F/} \times_{\operatorname{Fun}(K,J)} J$ at the cocartesian arrows by Lemma 1.1.2, which we precisely want to show is equivalent to a point.

To conclude, it suffices to remark that $\operatorname{colim}_{j\in J} \operatorname{Map}(x, j)$ is always contractible for any $x \in J$. Indeed, $\operatorname{Map}(x, -)$ is classified by the cocartesian fibration $J_{x/} \to J$ and $J_{x/}$ has an initial object, hence becomes contractible when inverting all its cocartesian edges. ☐

Let us also include a more pedestrian way of proving the last claim: note that there is a point in each $\operatorname{colim}_{j\in J} \operatorname{Map}(F(k), j)$ induced by $\operatorname{id}_{F(k)} : F(k) \to F(k)$. These lift to a point in the limit over K^{op} by functoriality and therefore we have a point $X \in \operatorname{colim}_{j\in J} \lim_{k\in K^{\mathrm{op}}} \operatorname{Map}(F(k), -)$. In consequence, since J is filtered and $*$ is compact, there is some $j \in J$ such that $X \in \lim_{k\in K^{\mathrm{op}}} \operatorname{Map}(F(k), j)$ and using the projection maps of the limits, this endows j with the structure of a cone point to F, i.e. there is a map $F \to \operatorname{cst}(j)$. In particular, we have found our category to be non-empty. Being more careful we could show that it is connected, and so forth to get the result. This precisely what the unstraightening captures in a rigorous manner.

Corollary 1.1.17 An object $X \in \mathcal{S}$ is compact if and only if it is a retract of a finite space.

Proof. Compact objects are closed under retracts: indeed if X is compact and A is a retract of X, then there is a diagram:

$$\begin{array}{ccc}
\operatorname{colim}_{i\in I} \operatorname{Map}(A, Z_i) & \longrightarrow & \operatorname{Map}(A, \operatorname{colim}_{i\in I} Z_i) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i\in I} \operatorname{Map}(X, Z_i) & \xrightarrow{\simeq} & \operatorname{Map}(X, \operatorname{colim}_{i\in I} Z_i) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i\in I} \operatorname{Map}(A, Z_i) & \longrightarrow & \operatorname{Map}(A, \operatorname{colim}_{i\in I} Z_i)
\end{array}$$

which exhibits the colimit-comparison map of A as a retract of an equivalence, hence an equivalence again.

Now, given a compact space X, write $X \simeq \operatorname{colim}_{i\in I} X_i$ with I filtered and the X_i finite. Then, $\mathrm{id}_X : X \longrightarrow X \simeq \operatorname{colim}_{i\in I} X_i$ must factor through one of the X_i. This exhibits X as a retract of X_i which concludes. □

Warning 1.1.18 Retracts of finite spaces need not be finite again. In general, retracts of finite spaces are called *finitely-dominated*. A Theorem of Wall, called Wall's finiteness obstruction, and related to Thomason's classification theorem shows that for a finitely-dominated space X, there is a class in $\widetilde{K_0}(\mathbb{S}[\Omega X])$ which vanishes if and only if X is finite.

Corollary 1.1.12 admits the following strengthening; in fact let us note that even if we present the result and the proof for $\mathcal{S}$, it holds *mutatis mutandis* when replacing $\mathcal{S}$ by $\mathcal{P}(\mathcal{C})$ for some small $\mathcal{C}$ and $\mathcal{S}^{\mathrm{fin}}$ by the full subcategory of $\mathcal{P}(\mathcal{C})$ containing the image of the Yoneda and stable under finite colimits.

Proposition 1.1.19 Suppose $\mathcal{C}$ has filtered-colimits, and write $i : \mathcal{S}^{\mathrm{fin}} \to \mathcal{S}$ for the inclusion of the full subcategory spanned by finite spaces. Then,

$$i^* : \mathrm{Fun}^{\omega}(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C})$$

is an equivalence, where the superscript ω denotes the full subcategory of *finitary*, i.e. filtered-colimit preserving functors. Its inverse is given by left Kan extension along i.

Proof. By Corollary 1.1.12, the above functor is conservative. We check that it has a left adjoint which is fully-faithful. In fact, we claim that this left adjoint is simply given by left Kan extension along i, which is automatically fully-faithful since i is. This follows from checking that if $f : \mathcal{S}^{\mathrm{fin}} \to \mathcal{C}$ is any functor, then $i_! F : \mathcal{S} \to \mathcal{C}$ preserves filtered colimits.

We remark that the functor $j_! i : \mathrm{Fun}((\mathcal{S}^{\mathrm{fin}})^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$, obtained by left Kan extending i along the Yoneda embedding of $\mathcal{S}^{\mathrm{fin}}$, has a filtered-colimit preserving right adjoint. This right adjoint is given by the formula

$$X \in \mathcal{S} \longmapsto \mathrm{map}(i(-), X) \in \mathrm{Fun}((\mathcal{S}^{\mathrm{fin}})^{\mathrm{op}}, \mathcal{S})$$

Note that filtered colimits in $\mathrm{Fun}((\mathcal{S}^{\mathrm{fin}})^{\mathrm{op}}, \mathcal{S})$ are computed pointwise so that the above formula shows the right adjoint commutes with filtered colimits precisely because finite spaces are compact in $\mathcal{S}$ by Proposition 1.1.15.

Now note that $i \simeq j_! i \circ j$ by fully-faithfulness of the Yoneda embedding j, so that we can perform the left Kan extension in two steps: first do $j_!$ and then $(j_! i)_!$ which is equivalently given by precomposition along the previous right adjoint. In particular, to conclude it suffices to check that $j_!$ lands in filtered-colimit preserving functors — this follows from Lemma 1.1.5. □

One can do a version of the above adapted to a regular cardinal $\kappa \geq \omega$.

Definition 1.1.20 A category is said to be κ-small if it is given by a simplicial set with a κ-small set of non-degenerate simplices.

The dependence in κ is as follows: if $\kappa \leq \lambda$, then every κ-small category is in particular λ-small.

Definition 1.1.21 A category J is said to be κ-filtered if for every κ-small category $\mathcal{C}$, the functor $\mathrm{cst} : J \to \mathrm{Fun}(\mathcal{C}, J)$ is cofinal.

It follows from the above that the dependency in κ is that if $\kappa \leq \lambda$, then every κ-small filtered category is λ-filtered. In particular, preserving λ-filtered colimits is a *weaker* condition that preserving κ-filtered ones.

Proposition 1.1.22 In $\mathcal{S}$, κ-filtered colimits commute with κ-small limits. Moreover, a category J is κ-filtered if and only if $\operatorname{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves κ-small limits.

Proof. Recall that a functor preserves κ-small limits if it preserves κ-small products and pullbacks. In particular, in light of Proposition 1.1.15, the first claim reduces to proving that κ-filtered colimits commute with κ-small products of spaces.

By Proposition 1.1.15, all the spheres S^n are compact; moreover, $\pi_0 : \mathcal{S} \to \mathrm{Set}$ preserves colimits and arbitrary products[1]. Since κ-filtered colimits are in particular filtered, this reduces the claim to a purely set-theoretical one:

$$\operatorname*{colim}_J \prod_{k\in K} X_{k,j} \longrightarrow \prod_{k\in K} \operatorname*{colim}_J X_{k,j}$$

for $X : J \times K \to \mathrm{Set}$ (with K discrete). This is true and checkable by hand.

In particular, this argument gives one direction of the claimed equivalence and we can run the same proof as in Corollary 1.1.16 (which we will explain later in its correct generality) to get the other implication. □

Definition 1.1.23 An object $X \in \mathcal{S}$ is κ-compact if and only if $\mathrm{Map}(X, -) : \mathcal{S} \to \mathcal{S}$ preserves κ-filtered colimits.

Remark 1.1.24 Unlike in the case $\kappa = \omega$, if κ is uncountable, a space is κ-compact if and only if it is κ-small. The proof starts in the same way: every κ-compact space is a retract of a κ-small space by the same arguments as Corollary 1.1.17, but splitting a retract is a countable colimit since Idem is countably small, hence κ-small spaces are stable under retracts, which concludes.

We also have the following generalization of Proposition 1.1.19:

Proposition 1.1.25 Write $i_\kappa : \mathcal{S}^\kappa$ for the inclusion of full subcategory of κ-small spaces. Suppose $\mathcal{C}$ has κ-filtered colimits, then,

$$i_\kappa^* : \mathrm{Fun}^\kappa(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{S}^\kappa, \mathcal{C})$$

is an equivalence, where the superscript κ denotes the full subcategory of κ-*finitary*, i.e. κ-filtered colimit preserving functors. Its inverse is given by left Kan extension along i.

Proof. We will prove a more general statement in the next section, but the reader is encouraged to adapt the arguments of Proposition 1.1.19. □

The above was a strengthening of the filteredness conditions, but one can also weaken the condition of being filtered as follows:

Definition 1.1.26 A category I is *sifted* if the functor $\mathrm{cst} : I \to \mathrm{Fun}(X, I)$ is cofinal for every finite *set* X.

■ **Example 1.1.27** Every filtered category is sifted. It is a well-known fact that $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal (see [Lur18, Tag 02QP], or play the combinatorical game through Quillen's Theorem A yourself) and since Δ is non-empty, Δ^{op} is sifted. Note that it is not filtered in general. ■

It holds that a functor preserves sifted colimits if and only if it preserves filtered colimits as well as geometric realization, i.e. Δ^{op}-indexed colimits. Note that the situation is different than if one defined this notion in the 1-categorical world; in particular, $\Delta_{\leq 1}$, which models the shape of a reflexive coequalizer, is sifted in the 1-categorical world but not in the higher sense.

Proposition 1.1.28 In $\mathcal{S}$, sifted colimits commute with finite products. Moreover, a category J is sifted if and only if $\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ commutes with finite products.

Proof. We already know that filtered colimits commute with finite products, hence it suffices to show that geometric realizations do. As we are not aware of a trick for this, we omit this proof — one strategy is to resolve the geometric realization in a model category of choice and prove it there (see Remark 5.5.8.12 and Lemma 6.1.3.14 of [Lur08]). The other direction of the equivalence will be proven more generally in the next section, and is the same as in Corollary 1.1.16. □

[1] The proof is as follows: show that it holds for Kan complexes, as in this MSE question and then use the fact that the localisation $\mathrm{Kan} \to \mathcal{S}$ preserves small products by virtue of the model structure, in fact only half of it suffices by [Cis19, Proposition 7.7.1]

Usually, an object $X \in \mathcal{C}$ such that $\mathrm{Map}(X, -)$ commutes with sifted colimits is called *compact projective*. In the case of spaces however, there is a much more usual name: *finite sets*.

Proposition 1.1.29 The full subcategory of $\mathcal{S}$ of those X such that $\mathrm{Map}(X, -)$ commutes with sifted colimits is the category FinSet of finite discrete spaces (i.e. sets). Moreover, the inclusion $i : \mathrm{FinSet} \to \mathcal{S}$ induces, for every $\mathcal{C}$ with sifted colimits, an equivalence

$$i^* : \mathrm{Fun}^{\mathrm{sft}}(\mathcal{S}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{FinSet}, \mathcal{C})$$

where the superscript sft denotes the full subcategory of sifted colimit preserving functors.

Proof. The second part will be subsumed in the next section. Let us only prove that FinSet is the claimed category: since $*$ is compact projective, so is every finite coproduct of it by virtue of Proposition 1.1.28.

Conversely, if X is a space such that $\mathrm{Map}(X, -)$ commutes with sifted colimits, we can find a Kan complex model for X itself and therefore realize it as the geometric realization of its n-simplices, which are filtered colimits of finite sets. In particular, there is a sifted colimit of finite sets whose colimit is X.

Therefore, id_X must factor through a finite set and to conclude, we note that the retract of a finite set is necessarily discrete and with finite π_0. □

1.2 Doctrines and colimit-completions

In Proposition 1.1.6, we explain how to freely add colimits to a category. But later throughout the section, we realized that we could also have added less colimits to a bigger category than $*$ and this was still “free” in some sense. In this section, we first explore how to freely add a class of colimits to a category while preserving some or in fact even, forcing a collection of cocones to be colimits. Afterwards, we explore the interaction between adding a shape of colimits freely and adding all colimits while respecting a shape, generalizing Propositions 1.1.19, 1.1.25 and 1.1.29.

Given a collection $S : \{f_\alpha : X_\alpha \to Y_\alpha\}$ of arrows in a category $\mathcal{C}$, we can ask whether a given object Z is S*-local*, i.e. if for every α, the natural map

$$f_\alpha^* : \mathrm{Nat}(Y_\alpha, Z) \longrightarrow \mathrm{Nat}(X_\alpha, Z)$$

is an equivalence. The collection of S-local objects is closed under limits. Note also that we can always saturate a collection of arrows S, i.e. add to S all the morphisms $X_\beta \to Y_\beta$ such that the above precomposition map is an equivalence for S-local objects, and this new collection $\overline{S}$ has the same local objects. Moreover, $\overline{S}$ automatically contains equivalences, is closed under 2-out-of-3 and is closed under colimits; the following is Proposition 6.2.3.12 [Lur18, Tag 04KG] — we will not reprove it.

Lemma 1.2.1 Suppose that S is a saturated class such that for every $X \in \mathcal{C}$, there is a map $f : X \to Y$ with Y S-local and $f \in S$. Then, the full subcategory $S^{-1}\mathcal{C}$ of S-local objects of $\mathcal{C}$ forms a reflexive subcategory, i.e. the inclusion has a left adjoint $L : \mathcal{C} \to S^{-1}\mathcal{C}$.

Let $\mathcal{K}$ be a collection of "shapes" (i.e. categories) which will serve as indexing our diagrams and that we will typically denote K and fix $\mathcal{C}$ some category. Our goal is to freely adding $\mathcal{K}$-shaped colimits, by this we mean K-colimits for every $K \in \mathcal{K}$ — in fact, we will sometimes need to be a bit more subtle and preserve some colimit diagrams in $\mathcal{C}$.

We let $\mathcal{R} = \{f_\alpha : K_\alpha^{\triangleright} \to \mathcal{C}\}$ be a collection of diagrams in $\mathcal{C}$, where $K_\alpha \in \mathcal{K}$ and $K_\alpha^{\triangleright}$ is our notation for freely adding a cocone point to K_α. In human language, we have chosen a collection of $\mathcal{K}$-shaped diagrams in $\mathcal{C}$ *and* a cocone for each of them.

Note the following two points, which are more technicalities than anything: first, we do not require these cocones to be colimit cocones so that we are doing something more general than also preserving some colimits, we are actually enforcing some diagrams to be colimit diagrams. Second, we do require that the K_α are in $\mathcal{K}$ which that if one wants to add say filtered colimits while preserving some cocartesian squares in a category which does not have all pushouts, the resulting category will have all pushouts.

Theorem 1.2.2 There is a category $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ with $\mathcal{K}$-shaped colimits and a functor $\gamma : \mathcal{C} \to \mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ which sends every diagram in $\mathcal{R}$ to a colimit diagram. Moreover, for every category $\mathcal{D}$ with $\mathcal{K}$-shaped colimits, precomposition by γ induces an equivalence:

$$\gamma^* : \mathrm{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{R}}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}_{\mathcal{K}}$ denote the full subcategory of functors preserving $\mathcal{K}$-shaped colimits and $\mathrm{Fun}_{\mathcal{R}}$ the full subcategory of those functors sending every diagram in $\mathcal{R}$ to a colimit diagram.

Moreover, if the diagrams in $\mathcal{R}$ are already colimit diagrams in $\mathcal{C}$, γ is fully-faithful.

Proof. We follow essentially the proof of [Lur08, Proposition 5.3.6.2]. Write $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ for the Yoneda embedding and consider the collection S

$$\operatorname*{colim}_{K_\alpha}(j \circ f_\alpha) \mid_{K_\alpha} \longrightarrow (j \circ f)(*_{K_\alpha}))$$

for each diagram $f_\alpha : K_\alpha^{\triangleright} \to \mathcal{C}$ in $\mathcal{R}$ where $*_{K_\alpha}$ denotes the cocone point of $K_\alpha^{\triangleright}$. We first check the conditions of Lemma 1.2.1. A S-local object ϕ for the above collection is simply a presheaf $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ such that $\phi \circ f_\alpha$ is a limit diagram. We note that given a presheaf ϕ, there is an initial map

$$\eta : \phi \longrightarrow \psi$$

where the right hand side is S-local, and we can simply take ψ to be the limit indexed by the full subcategory $\mathcal{C}_{\psi/}$ spanned by those maps whose target is S-local. But now, for ξ another S-local object, the map

$$\mathrm{Nat}(\psi, \xi) \xrightarrow{\simeq} \mathrm{Nat}(\phi, \xi)$$

is necessarily an equivalence by the universal property of η. Note that ψ is just the value at ψ of the right Kan extension of the restriction to the full subcategory of S-local objects, so we could have done away with the lemma and simply produce the adjoint directly.

We let $L : \mathcal{C} \to S^{-1}\mathcal{C}$ be the left adjoint to the category of S-local objects, which exists by Lemma 1.2.1 and we write $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ for the smallest full subcategory of $S^{-1}\mathcal{C}$ containing the image of $L \circ j$ and stable under $\mathcal{K}$-shaped colimits. We claim that for every $\mathcal{D}$ with $\mathcal{K}$-shaped colimits, precomposition along $L \circ j$ induces an equivalence:

$$(L \circ j)^* : \mathrm{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{R}}(\mathcal{C}, \mathcal{D})$$

We first note that $L \circ j$ sends the diagrams of $\mathcal{R}$ to colimits since L inverts the maps in S and preserves colimits, so the above functor is well-defined. Moreover, the minimality hypothesis on $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ implies that this functor is conservative.

Suppose for a moment that $\mathcal{D}$ has all small colimits. This extra-assumption allows the left Kan extension along $L \circ j$ to exists and it is given by $F \mapsto (j_! F) \circ i$ where $i : \mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}) \to \mathcal{C}$ is the inclusion. In particular, since i preserves $\mathcal{K}$-shaped colimits, and $j_! F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ all of them by Lemma 1.1.5, the left Kan extension $(L \circ j)_!$ does restrict to the wanted categories, so that it is a left adjoint to $(L \circ j)^*$.

Now given $F : \mathcal{C} \to \mathcal{D}$ that sends $\mathcal{R}$ to colimits, we want to check that the map

$$F \longrightarrow (L \circ j)^* (L \circ j)_! F$$

is an equivalence. Now note that because j is fully-faithful, it will suffice to prove that $j_! F \to L^* L_! j_! F$ is an equivalence. But because F sends every f_α to a colimit already, the functor $j_! F$ factors through the category of S-local objects $S^{-1}\mathcal{P}(\mathcal{C})$ (in fact this is an if and only if), and therefore is canonically equivalent to $L_* L_! j_! F$.

Finally, we reduce to the case where $\mathcal{D}$ has small colimits. Note that $\mathcal{D} \to \overline{\mathcal{D}} := \mathrm{Fun}(\mathcal{D}, \mathcal{S})^{\mathrm{op}}$ is colimit-preserving, as the opposite of the Yoneda embedding of $\mathcal{D}^{\mathrm{op}}$, and its target has all small colimits since $\mathcal{S}$ has small limits. Hence, the above applies to $\overline{\mathcal{D}}$ and it suffices to show that the following square is cartesian:

$$\begin{array}{ccc} \mathrm{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}),\mathcal{D}) & \longrightarrow & \mathrm{Fun}_{\mathcal{R}}(\mathcal{C},\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}),\overline{\mathcal{D}}) & \longrightarrow & \mathrm{Fun}_{\mathcal{R}}(\mathcal{C},\overline{\mathcal{D}}) \end{array}$$

In turn, this means showing that if $F : \mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}) \to \overline{\mathcal{D}}$ preserves $\mathcal{K}$-shaped colimits and restricts to $\mathcal{D}$ along α, then F itself was already landing in $\mathcal{D}$. But the full subcategory $F^{-1}(\mathcal{D})$ contains $\mathcal{C}$ by assumption and is closed under $\mathcal{K}$-shaped colimits, hence $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}) \subset F^{-1}(\mathcal{D})$ which concludes by minimality.

It remains to explain the last claim of the Theorem: suppose that every diagram in $\mathcal{R}$ is a colimit diagram. Then, to show that α is fully-faithful, it suffices to check that $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ lands in S-local objects for the aforementioned collection of maps S. We have to show for every $X \in \mathcal{C}$

$$\mathrm{Nat}(j(F(*_R)), j(X)) \xrightarrow{\simeq} \mathrm{Nat}(\operatorname*{colim}_{R} j \circ F\mid_R, j(X))$$

for every diagram $F : \overline{R} \to \mathcal{C}$. Using the Yoneda lemma and the fact that $F(*_R) \simeq \mathrm{colim}_R F\mid_R$, this is clearly follows from the fact that

$$\mathrm{Map}(\operatorname*{colim}_{r\in R} F(r), X) \longrightarrow \lim_{r \in R} \mathrm{Map}(F(r), X)$$

is an equivalence. □

We say that $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ is $\mathcal{C}$ to which we have added $\mathcal{K}$-shaped colimits while forcing $\mathcal{R}$ to be colimits. If $\mathcal{R}$ is already a collection of colimits cocones, we say that instead "while *preserving* $\mathcal{R}$-colimits". Finally, if $\mathcal{R} = \emptyset$, we say that we have *freely* added $\mathcal{K}$-shaped colimits to $\mathcal{C}$

■ **Example 1.2.3** If $\mathcal{K} = \mathbf{Cat}$ and $\mathcal{R} = \emptyset$, Proposition 1.1.6 has guaranteed that $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}) \simeq \mathcal{P}(\mathcal{C})$ and j is the Yoneda embedding.

We have also seen that $\mathcal{S}$ is the common value of $\mathcal{P}^{\omega-filt}_{\emptyset}(\mathcal{S}^{\mathrm{fin}})$, $\mathcal{P}^{\kappa-filt}_{\emptyset}(\mathcal{S}^{\kappa})$ and $\mathcal{P}^{sifted}_{\emptyset}(\mathrm{FinSet})$. In all three cases, we also got that the resulting category had all small colimits and the inclusion preserved the complementary type of them, i.e.

$$\mathcal{P}^{\omega-filt}_{\emptyset}(\mathcal{S}^{\mathrm{fin}}) \simeq \mathcal{P}^{small}_{fin}(\mathcal{S}^{\mathrm{fin}})$$
$$\mathcal{P}^{\kappa-filt}_{\emptyset}(\mathcal{S}^{\kappa}) \simeq \mathcal{P}^{small}_{\kappa-small}(\mathcal{S}^{\kappa})$$
$$\mathcal{P}^{sifted}_{\emptyset}(\mathrm{FinSet}) \simeq \mathcal{P}^{small}_{\coprod}(\mathrm{FinSet})$$

where we hope all of the super/subscripts are clear. This is not a coincidence, and we will explain it later in the section. ■

Remark 1.2.4 Given $\mathcal{K}$ and a choice $\mathcal{R}$ of cocones in $\mathcal{C}$, the association $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(-)$ naturally refines to a functor thanks to its universal property. Its source is the category of pairs $(\mathcal{C}, \mathcal{R})$ where maps are functors which preserve the collections of the chosen cocones[a]; its target is the category of categories with $\mathcal{K}$-shaped colimits.

In fact, $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(-)$ is a left adjoint to the functor sending $\mathcal{D}$ with $\mathcal{K}$-shaped colimits to $(\mathcal{D}, \{\mathcal{K}-$ shaped colimits cocones in $\mathcal{D}\}$. This is especially pratical because this adjunction often descends when considering subcategories of the form $(\mathcal{C}, \mathcal{R})$ with $\mathcal{R}$ chosen functorially (such as finite colimit cocones, etc ...).

[a]More precisely, $f : \mathcal{C} \to \mathcal{D}$ must be such that if $p : K^{\triangleright}_{\alpha} \to \mathcal{C}$ is in $\mathcal{R}_{\mathcal{C}}$, then $f \circ p$ is $\mathcal{R}_{\mathcal{D}}$

We now showcase three examples which will be important for us later on. The first one will be ubiquitous throughout this lecture.

Definition 1.2.5 The *κ-inductive completion* of $\mathcal{C}$, denoted $\mathrm{Ind}_{\kappa}(\mathcal{C})$, is the category obtained from $\mathcal{C}$ by freely adding κ-filtered colimits, i.e. $\mathrm{Ind}_{\kappa}(\mathcal{C}) \simeq \mathcal{P}^{\kappa-filt}_{\emptyset}(\mathcal{C})$.

The second kind of example will be a little less common for us, but does have its use in other parts of higher category theory:

Definition 1.2.6 The *non-abelian derived category* of $\mathcal{C}$, denoted $\mathcal{P}_\Sigma(\mathcal{C})$, is the category obtained from $\mathcal{C}$ by freely adding sifted colimits, i.e. $\mathcal{P}_\Sigma(\mathcal{C}) \simeq \mathcal{P}_\emptyset^{sifted}(\mathcal{C})$. In recent years, this category has also been known as the *animation* of $\mathcal{C}$.

We write Ret for the category generated by the graph two vertices A, X, with non-trivial maps $i : A \to X$, $r : X \to A$ such that $r \circ i = \mathrm{id}$. We write Idem for the full subcategory spanned by the vertex X. We note that the inclusion Idem $\to$ Ret is cofinal, hence every functor with source Ret is left Kan extended from Idem. Therefore, a functor $F : \mathrm{Idem} \to \mathcal{C}$ admits a colimit if and only if it extends to Ret.

Definition 1.2.7 We write $\mathrm{Idem}(\mathcal{C})$ for the category obtained from $\mathcal{C}$ by freely adding retracts to idempotent, i.e. $\mathrm{Idem}(\mathcal{C}) \simeq \mathcal{P}_\emptyset^{\mathrm{Idem}}(\mathcal{C})$.

Remark 1.2.8 If $\mathcal{C}$ admits finite colimits, it need not be that every idempotent has a colimit, i.e. Idem is *not* a finite category in the higher categorical world.

We finish this section by ideas of Rezk [Rez21], himself based on 1-categorical results notably by Adámek, Lawvere, Rosický and many of their collaborators, from which tries to encapsulate the fact that the three Proposition 1.1.15 1.1.22, 1.1.28 have similar statements and similar proofs (which we ourselves have skipped for this very reason).

For the rest of the section, fix $\mathcal{U}$ a collection of small categories which we call our *doctrine*.

Definition 1.2.9 A category J is called *$\mathcal{U}$-filtered* if $\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves $\mathcal{U}$-shaped limits.

A category J is called *weakly $\mathcal{U}$-filtered* if for every $U \in \mathcal{U}$, the functor $\mathrm{cst} : J \to \mathrm{Fun}(U^{\mathrm{op}}, J)$ is cofinal.

■ **Example 1.2.10** If J has all U^{op}-colimits for $U \in \mathcal{U}$, then J is weakly $\mathcal{U}$-filtered, since cst has a left adjoint. Note also that although it looks different, this definition recovers the one for κ-filtered using $\mathcal{U} = \{\kappa\text{-small categories}\}$ since those are closed under op. ■

The reason for the op appearing in our definition is so that they disappear in the following:

Lemma 1.2.11 A category J is weakly $\mathcal{U}$-filtered if and only if $\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves $\mathcal{U}$-shaped limits of corepresentable functors. In particular, $\mathcal{U}$-filtered categories are weakly $\mathcal{U}$-filtered.

Proof. By Quillen's Theorem A [Lur08, Theorem 4.1.3.1], J is weakly $\mathcal{U}$-filtered if and only if for every $U \in \mathcal{U}$ and every $F : U^{\mathrm{op}} \to J$, the category $\mathrm{Fun}(U^{\mathrm{op}}, J)_{F/} \times_{\mathrm{Fun}(U^{\mathrm{op}}, J)} J$ is contractible. We recall that this category is the total space of the unstraightening of $\mathrm{Nat}(F, \mathrm{cst}(j)) \simeq \lim_{u \in U} \mathrm{Map}(F(u), j)$ as a functor $\mathcal{J} \to \mathcal{S}$.

On the other hand, colim_J preserves U-limits of corepresentables if for every $F : U^{\mathrm{op}} \to J$ the map

$$\eta : \operatorname*{colim}_{j \in J} \lim_{u \in U} \mathrm{Map}(F(u), j) \longrightarrow \lim_{u \in U} \operatorname*{colim}_{j \in J} \mathrm{Map}(F(u), j)$$

is an equivalence.

Note that by Lemma 1.1.2 $\mathrm{colim}_{j \in J} \mathrm{Map}(F(u), j)$ is always contractible as $\mathrm{Map}(F(u), -)$ is classified by $J_{F(u)/} \to J$ whose total category has an initial object. In particular, if η is an equivalence then the left hand side is contractible and therefore again by Lemma 1.1.2, the category $\mathrm{Fun}(U^{\mathrm{op}}, J)_{F/} \times_{\mathrm{Fun}(U^{\mathrm{op}}, J)} J$ is weakly contractible. Reciprocally, if this category is weakly-contractible, then the left hand side is also contractible hence both sides are contractible and therefore the map is an equivalence. □

Warning 1.2.12 The converse need not hold. Rezk has some examples in section 6 of his paper. Let us also mention another. Consider $\mathcal{U} := \{\kappa\text{-small sets}\}$ for some $\kappa > \omega$, then we claim that the category Δ_κ of linearly order κ-small sets and order preserving maps is weakly $\mathcal{U}$-filtered; the argument adapts from the standard argument showing that Δ is (weakly) ω-sifted.

Nonetheless, it can be shown that κ-small products of spaces do not commute with Δ_κ-indexed colimits. In fact, the main result of [AKV00] shows that in the 1-categorical world, $\mathcal{U}$-filtered categories are actually κ-filtered (!), which Δ_κ is not.

Therefore, if J is such that $\operatorname{colim}_J : \operatorname{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves κ-small products, so will the colimit functor valued in Set since π_0 preserves colimits and arbitrary products[a], and therefore J will be κ-filtered. Since κ-small sets are in particular κ-small categories, we see that colim_J preserves κ-small products if and only if J is κ-filtered.

[a]Yet another thing we do not know of a good reference for, except for https://math.stackexchange.com/questions/713792/does-pi-0-preserve-infinite-productsthis MSE answer which is not quite written in a model-independant way.

Definition 1.2.13 We call a doctrine *sound* if the converse of Lemma 1.2.11 holds.

We will not draw an explicit criterion for soundness in this version of the course notes, but we want to include one eventually.

Given a doctrine $\mathcal{U}$, we write $\overline{\mathcal{U}}$ for the collection of U such that $\operatorname{colim}_J : \operatorname{Fun}(J, \mathcal{S}) \to \mathcal{S}$ preserves U-shaped limits for every $\mathcal{U}$-filtered J. By definition $\mathcal{U} \subset \overline{\mathcal{U}}$. Moreover, if $\mathcal{U} \subset \mathcal{V}$, then $\overline{\mathcal{U}} \subset \overline{\mathcal{V}}$.

Definition 1.2.14 A doctrine $\mathcal{U}$ is *regular* if $\mathcal{U} = \overline{\mathcal{U}}$.

Example 1.2.15 The doctrine of κ-small categories is regular as soon as κ is a regular cardinal. ■

The categories in $\overline{\emptyset}$ are often known are universal (co)limits. In particular, for every regular doctrine $\mathcal{U}$, we have $\overline{\emptyset} \subset \mathcal{U}$.

Lemma 1.2.16 The category Idem belongs to $\overline{\emptyset}$, i.e. it is preserved by $\operatorname{colim}_J : \operatorname{Fun}(J, \mathcal{S}) \to \mathcal{S}$ for every J.

Proof. This follows directly from the fact that $\mathrm{Idem} \to \mathcal{C}$ has a colimit if and only if it extends to Ret. □

Note that since $\mathrm{Idem}^{\mathrm{op}} \simeq \mathrm{Idem}$, the above statement also holds for limit preservation.

Definition 1.2.17 An object $X \in \mathcal{C}$ is $\mathcal{U}$-compact if $\operatorname{Map}(X, -) : \mathcal{C} \to \mathcal{S}$ commutes with $\mathcal{U}$-filtered colimits.

Example 1.2.18 The point $* \in \mathcal{S}$ is $\mathcal{U}$-compact for any collection $\mathcal{U}$. Any initial object is $\mathcal{U}$-compact for any $\mathcal{U}$. ■

Note that $\mathcal{U}$-compacity only depends on the class of $\mathcal{U}$-filtered categories, which itself only depends on the regular doctrine $\overline{\mathcal{U}}$ generated by $\mathcal{U}$.

Lemma 1.2.19 Let $\mathcal{C}$ be a category. The subcategory of $\mathcal{U}$-compact objects of $\mathcal{C}$ is closed under all the U^{op}-indexed colimits that exist, for $U \in \overline{\mathcal{U}}$. In particular, $\mathcal{U}$-compact objects are always closed under retracts.

Proof. By Lemma 1.2.16, $\mathrm{Idem} \in \overline{\emptyset} \subset \overline{\mathcal{U}}$. Hence, it suffices to prove the first claim. Given a U^{op}-indexed diagram of compact objects X_u which admits a colimit X in $\mathcal{C}$, we have $\operatorname{Map}(X, -) \simeq \lim_{u \in U} \operatorname{Map}(X_u, -)$. Now given J which is $\mathcal{U}$-filtered and a J-indexed diagram Y_j, the canonical map

$$\operatorname*{colim}_{j \in J} \lim_{u \in U} \operatorname{Map}(X_u, Y_j) \xrightarrow{\simeq} \lim_{u \in U} \operatorname*{colim}_{j \in J} \operatorname{Map}(X_u, Y_j)$$

is an equivalence. Using that each X_u is compact, the right hand side identifies with $\operatorname{Map}(X, \operatorname{colim}_j Y_j)$ whereas the left hand side is $\operatorname{colim}_j \operatorname{Map}(X, Y_j)$ which concludes. □

Another consequence of the above is that the full subcategory of $\mathcal{S}$ of $\mathcal{U}$-compact objects always contain the full subcategory generated by $*$ under U^{op}-indexed colimits for $U \in \overline{\mathcal{U}}$.

Remark 1.2.20 Actually, the previous proof only used that colim_J commutes with corepresentable presheaves, so it also applies to a version of compact objects defined with respect to weakly $\mathcal{U}$-filtered colimits.

Definition 1.2.21 Let $\mathcal{C}$ be a small category. We write $\operatorname{Ind}_{\mathcal{U}}(\mathcal{C})$ for the category obtained from $\mathcal{C}$ by freely $\mathcal{U}$-filtered colimits, i.e. $\operatorname{Ind}_{\mathcal{U}}(\mathcal{C}) \simeq \mathcal{P}_{\emptyset}^{\mathcal{U}-\mathrm{filt}}(\mathcal{C})$ in the notations of Theorem 1.2.2.

■ **Example 1.2.22** It is standard to write $\operatorname{Ind}(\mathcal{C})$ for $\operatorname{Ind}_{\omega-filt}(\mathcal{C})$, i.e. freely adding filtered colimits to a category, and call it the *inductive completion of* $\mathcal{C}$ (in the sense of objects in $\operatorname{Ind}(\mathcal{C})$ being formal inductive systems, inductive being a old (possibly weaker?) name for filtered.

More generally, we will write $\operatorname{Ind}_{\kappa}(\mathcal{C})$ for freely adding κ-filtered colimits to a category. ■

■ **Example 1.2.23** The "old-school" name and notation for $\operatorname{Ind}_{sifted}(\mathcal{C})$, freely adding sifted colimits (equivalently, filtered colimits and geometric realizations) is $\mathcal{P}_{\Sigma}$, the *non-abelian derived category.* If $\mathcal{C}$ has finite coproducts, the following theorem will show that it also coincides with finite-product preserving presheaves on spaces, a process often call *animation* under the influence of the condensed mathematics crowd, which call $\mathcal{S}$ the category of anima.

In particular, as a consequence of what we explained in the previous section, $\mathcal{S}$ is the animation of the finite coproduct closure of $*$, i.e. the category FinSet of finite sets. ■

Theorem 1.2.24 — Rezk. Let $\mathcal{U}$ be a sound doctrine. Suppose $\mathcal{C}$ is a category with U^{op}-colimits for every $U \in \mathcal{U}$, then there is an equivalence

$$\operatorname{Ind}_{\mathcal{U}}(\mathcal{C}) \simeq \operatorname{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

where $\operatorname{Fun}_{\mathcal{U}-\lim}$ designates the full subcategory of $\mathcal{U}$-limit preserving presheaves.

Moreover, $\operatorname{Ind}_{\mathcal{U}}(\mathcal{C})$ has U^{op}-colimits for every $U \in \mathcal{U}$ and they are preserved by the fully-faithful $j_{\mathcal{U}} : \mathcal{C} \to \operatorname{Ind}_{\mathcal{U}}(\mathcal{C})$, so there is another equivalence $\operatorname{Ind}_{\mathcal{U}}(\mathcal{C}) \simeq \mathcal{P}_{\mathcal{U}^{\mathrm{op}}}^{\mathcal{U}-\mathrm{filt}}(\mathcal{C})$.

Proof. We begin by reducing the second assertion to the first; for this, we prove that the category $\operatorname{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ has U^{op}-colimits for $U \in \mathcal{U}$ and they are preserved by the Yoneda embedding. It follows from Lemma 1.2.1 that the inclusion

$$\operatorname{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \longrightarrow \mathcal{P}(\mathcal{C})$$

has a left adjoint. Indeed, by the Yoneda lemma, we see that the left hand side is the category of presheaves which are local with respect to the collection of maps

$$\operatorname*{colim}_{u \in U^{\mathrm{op}}} j(X^{\mathrm{op}}(u)) \longrightarrow j(\operatorname*{colim}_{u \in U^{\mathrm{op}}} X^{\mathrm{op}}(u))$$

for every $X : U \to \mathcal{C}^{\mathrm{op}}$, and this collection satisfies the hypotheses of 1.2.1. In particular, since $\mathcal{P}(\mathcal{C})$ has colimits, so does $\operatorname{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and the above collection of maps makes it clear that U^{op}-colimits are preserved by j.

We now check that $\operatorname{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ has the wanted universal property. In fact, we check that it coincides with the description of $\mathcal{P}_{\emptyset}^{\mathcal{U}-\mathrm{filt}}(\mathcal{C})$ given by Theorem 1.2.2. Note that since our $\mathcal{R}$ is empty, this is precisely the smallest full subcategory of $\mathcal{P}(\mathcal{C})$ containing the image of the Yoneda and closed under $\mathcal{U}$-filtered colimits.

We first remark that if $X \in \mathcal{C}$, then $j(X) := \operatorname{Map}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ preserves $\mathcal{U}$-limits. Moreover, $\operatorname{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is closed under $\mathcal{U}$-filtered colimits in $\mathcal{P}(\mathcal{C})$ precisely because those commute with $\mathcal{U}$-limits in spaces. To conclude, it suffices to show that every presheaf $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ which commute with $\mathcal{U}$-limits is a $\mathcal{U}$-filtered colimit of representables.

We have already proven this statement: indeed, recall from the proof of Proposition 1.1.6 that the canonical map

$$\operatorname*{colim}_{j(Y) \to \phi} \operatorname{Map}(X, Y) \longrightarrow \phi(X)$$

is an equivalence. We claim that the category $\operatorname{Un}^{cart}(\phi) \simeq \mathcal{P}(\mathcal{C})_{/\phi} \times_{\mathcal{P}(\mathcal{C})} \mathcal{C}$ which indexes the colimit is $\mathcal{U}$-filtered, i.e. that space-valued colimits indexed by $\operatorname{Un}^{cart}(\phi)$ commute with $\mathcal{U}$-limits. Note that it is automatically weakly $\mathcal{U}$-filtered since it admits U^{op}-colimits; this is straightforward from identifying $\mathrm{Un}^{cart}(\phi) \simeq \mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\phi} \times_{\mathrm{Fun}_{\mathcal{U}-\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})} \mathcal{C}$ and the first statement we proved. ☐

Remark 1.2.25 We found throughout the proof that another description of $\mathrm{Ind}_{\mathcal{U}}(\mathcal{C})$ is the category of presheaves ϕ such that $\mathrm{Un}(\phi)$ is a $\mathcal{U}$-filtered category. In fact, the above shows that this description stands even if $\mathcal{C}$ has not enough $\mathcal{U}^{\mathrm{op}}$-shaped colimits and $\mathcal{U}$ is not necessarily sound. We refer to Rezk's manuscript [Rez21] for a more general picture (including what happens when $\mathcal{U}$ is not sound).

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