Worksheet 1 - Homotopy II

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Exercise 1 Two-out-of-three/six

Let $\mathcal{W}$ be a collection of arrow of $\mathcal{C}$, we say $\mathcal{W}$ satisfies 2-out-of-3 if for every couple of composable arrows f, g, any two out of the three $f, g, g \circ f$ being in $\mathcal{W}$ implies the third is. We say $\mathcal{W}$ satisfies 2-out-of-6 if for every triple of composable arrows f, g, h such that $g \circ f \in \mathcal{W}$ and $h \circ g \in \mathcal{W}$, then the four other maps $f, g, h, h \circ g \circ f \in \mathcal{W}$.

1. Let $\mathcal{W}$ be a collection of arrow of $\mathcal{C}$ satifying 2-out-of-6. Show $\mathcal{W}$ satisfies 2-out-of-3.
2. a) Show the collection of isomorphisms of a category always satisfies 2-out-of-3. Does it also satisfies 2-out-of-6?
   b) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\mathcal{I}_{\mathcal{C}}$ be the collection of isomorphisms of $\mathcal{C}$. Show $F(\mathcal{I}_{\mathcal{C}})$ satisfies 2-out-of-6.
   c) Find a collection of arrows satisfying 2-out-of-3 but not 2-out-of-6.
3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\mathcal{W}$ be a collection of arrows of $\mathcal{D}$ satisfying 2-out-of-3 (resp. 2-out-of-6). Show $F^{-1}(\mathcal{W})$ satisfies 2-out-of-3 (resp. 2-out-of-6).
4. Let $\mathcal{W}$ be a collection of arrows in $\mathcal{C}$, and denote $L : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ the localisation functor. Show that $\mathcal{C}[\mathcal{W}^{-1}] \simeq \mathcal{C}[L^{-1}(\mathrm{Iso})^{-1}]$.
5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\mathcal{W}$ be a collection of arrows of $\mathcal{C}$ satisfying 2-out-of-3. Is it true that $F(\mathcal{W})$ satisfies 2-out-of-3 ?

Exercise 2 Lifting Properties

In the following, $\mathcal{C}$ is a category with small limits and colimits. Let $i : A \to B$ and $p : X \to Y$ be two morphisms, we write $i \perp p$ ("i has the LLP against p" or equivalently, "p has the RLP against i") if for every square as follows, there exists a dotted arrow making the diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

If $\mathcal{W}$ is a collection of morphisms in $\mathcal{C}$, we let ${}^{\perp}\mathcal{W}$ (or $LLP(\mathcal{W})$) be the collection of arrows i such that for every $p \in \mathcal{W}$, $i \perp p$. Conversly, $\mathcal{W}^{\perp}$ (or $RLP(\mathcal{W})$) is the collection of p such that for every $i \in \mathcal{W}$, $i \perp p$.

1. Let $\mathcal{W}$ be a collection of morphisms of $\mathcal{C}$.
   a) Show that $LLP(\mathcal{W})$ and $RLP(\mathcal{W})$ contain every isomorphism and are stable under composition.
   b) Show that $LLP(\mathcal{W})$ is stable under cobase change (i.e. if $i : A \to B \in LLP(\mathcal{W})$ and $f : A \to A'$ is any map, then $A' \to A' \coprod_A B \in LLP(\mathcal{W})$).
   c) Show that $RLP(\mathcal{W})$ is stable under base change.
2. Let $f : X \to Y$ be a map of $\mathcal{C}$. Show that $LLP(\{f\})$ is the collection of every arrow in $\mathcal{C}$ if and only if f has a left inverse (resp. $RLP(\{f\})$ and right invertible).

(Cisinski) A class of arrows $\mathcal{W}$ is said to be stable under *transfinite composition* if, for every well-ordered set I with initial object $\emptyset$ and every functor $F : I \to \mathcal{C}$ such that every partial colimit $F_{<i} := \mathrm{colim}_{j<i} F(j)$ exists and every map $F_{<i} \to F(i)$ is in $\mathcal{W}$, then $\mathrm{colim}\, F$ exists and the map $F(0) \to F_{\leq i}$ is in $\mathcal{W}$.

3. Show that $LLP(\mathcal{W})$ and $RLP(\mathcal{W})$ are stable under transfinite composition.

Exercise 3 Retracts and models

Let $f : A \to B$ and $g : X \to Y$. We say that f *is a retract of* g if there exists a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{p} & A \\ \downarrow f & & \downarrow g & & \downarrow f \\ B & \xrightarrow{j} & Y & \xrightarrow{q} & B \end{array}$$

such that the composites pi and qj are the respective identities id_A and id_B.

1. *(The Retract Arguement)* Let $i : A \to B$ and $p : B \to C$ and denote $f := pi$. Suppose $f \in LLP(\{p\})$, then show that f is a retract of i
2. Let $\mathcal{A}$ be a (co)complete category and $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ three collections of arrows such that:
   1. $\mathcal{W}$ is closed under 2-out-of-6.
   2. $\mathcal{C}$, $\mathcal{F}$, $\mathcal{W}$ are closed under retracts.
   3. $\mathcal{C} \cap \mathcal{W} \subset LLP(\mathcal{F})$ and $\mathcal{C} \subset LLP(\mathcal{F} \cap \mathcal{W})$
   4. Every arrow of $\mathcal{A}$ factors as an arrow of $\mathcal{C}$ followed by an arrow of $\mathcal{F}$, and both can be taken acyclic (i.e. also in $\mathcal{W}$) but not necessarily simultaneously.

   a) Using the first question, show that $\mathcal{C} \cap \mathcal{W} = LLP(\mathcal{F})$ and $\mathcal{C} = LLP(\mathcal{F} \cap \mathcal{W})$
   b) Deduce that $\mathcal{F} \cap \mathcal{W} = RLP(\mathcal{C})$ and $\mathcal{F} = RLP(\mathcal{C} \cap \mathcal{W})$.
   c) Show that $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ describes a model structure on $\mathcal{A}$.

Exercise 4 Miscellaneous about colimits

For simplicity, all rings are commutative.

1. Let R be a commutative ring. Show that $A \otimes_C B \simeq A \coprod_C B$ in the category of R-algebras.
2. Let $f, g : A \to B$ be two morphisms of a category $\mathcal{C}$. The coequalizer of f, g is the map $h : B \to X$ given by the following universal property: for every $\alpha : B \to C$ such that $\alpha \circ f \simeq \alpha \circ g$, α factors through h:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[rr, bend left, "\alpha"] & \mathrm{CoEq}(f,g) \arrow[r, dotted] & C \end{tikzcd}$$

   a) Compute the coequalizer of two morphisms in Set. Is it different from the pushout $B \coprod_A B$ of f and g?
   b) Suppose $\mathcal{C}$ admits a zero object. Show the coequalizer of f and 0 is the kernel of f.
   c) Let $\mathcal{A}$ be an abelian category. Show that $\mathrm{CoEq}(f, g) \simeq \ker(f - g)$.
3. Let R be a ring, and $t \in R$. We denote $\phi_t : R \to R$ the morphism $a \mapsto at$. Compute the colimit of the following diagram in the category of rings:

$$R \xrightarrow{\phi_t} R \xrightarrow{\phi_t} R \xrightarrow{\phi_t} R \xrightarrow{\phi_t} ...$$

   Is the colimit different in the category of R-algebras?
4. Show that a functor L admitting a left adjoint preserves colimits.

Exercise 5 Finitely presented and compact modules

Let R be a commutative ring. A R-module M is said to be finitely presented if there exists a surjection $R^k \twoheadrightarrow M$ with finitely generated kernel.

1. What are the finitely presented k-vector spaces, when k is a field?
2. Let $F, G : \mathrm{Mod}_R \to \mathcal{C}$ be two left-exact functors equipped with a natural transformation $\eta : F \implies G$. Suppose $\eta(R^k)$ is an isomorphism for every $k \in \mathbb{N}$. Deduce that η is an isomorphism on every finitely presented module.
3. a) Show any R-module can be written as the filtered colimit of finitely presented modules.
   b) Show any direct summand of a finitely-presented module is itself finitely-presented.

c) Deduce that any compact R-module is finitely presented.
d) Using the previous question, show that in fact, M is compact in the category of R-modules if and only if it is finitely presented.

Exercise 6 Ken Brown's Lemma

Let $\mathcal{C}$ be a model category and $\mathcal{D}$ a category with weak equivalences satisfying 2-out-of-3. Suppose $F : \mathcal{C} \to \mathcal{D}$ takes trivial cofibrations between cofibrant objects to weak equivalences. Show that F takes in fact all weak equivalences between cofibrant objects to weak equivalences.

Exercise 7 Unique Lifts and Orthogonal Factorizations (*Zhen Lin on MSE*)

Let i and p be two arrows such that $i \perp p$. We say that i and p are *orthogonal* if the solution to every lifting problem is unique (i.e. the dotted arrow making a given square commute is always unique). We let $(\mathcal{A}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model category.

1. Suppose $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}$ are orthogonal to one another. Show that trivial cofibrations between fibrant objects are isomorphisms and every morphism between fibrant objects is a fibration.
2. Deduce that if both $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}$ as well as $\mathcal{F} \cap \mathcal{W}$ and $\mathcal{C}$ are orthogonal to one another, then every weak equivalence between cofibrant-fibrant objects is an isomorphism. In particular, $\mathcal{A}[\mathcal{W}^{-1}] \simeq \mathcal{A}_{cf}$ where $\mathcal{A}_{cf}$ is the full subcategory of cofibrant-fibrant objects.

Exercise 8 Slice model categories

Let $X \in \mathcal{A}$ and $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{A}$. We denote $\mathcal{A}_{/X}$ the category whose objects are maps $\alpha : Y \to X$ of $\mathcal{A}$ and whose morphisms are commutative triangles

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "f"'] & X \\ Y' \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

Similarly, we denote $\mathcal{C}_{/X}$ (resp. $\mathcal{F}_{/X}$, $\mathcal{W}_{/X}$) the morphisms of $\mathcal{A}_{/X}$ as above where $f \in \mathcal{C}$ (resp.).

1. Show that $(\mathcal{C}_{/X}, \mathcal{F}_{/X}, \mathcal{W}_{/X})$ determines a model structure on $\mathcal{A}_{/X}$. We call it the *slice model structure*.
2. What are the fibrant objects on the above described model structure ? The cofibrant objects ?

Exercise 9 A model structure on Cat (*Charles Rezk*)

Denote Cat the category of small categories. We assume that it is complete and cocomplete. We let $\mathcal{W}$ denote equivalences of categories and $\mathcal{C}$ denote functors that are injective on objects. We let $\mathcal{F}$ denote functors $F : \mathcal{A} \to \mathcal{B}$ such that for every isomorphism $g : F(a) \to b$ of $\mathcal{B}$, there is a map $f : a \to a'$ with $g = F(f)$; such functors are called *isofibrations*.

1. Denote $*$ the category with one object and no non-trivial arrows, and I the category with two objects $0, 1$ and exactly one isomorphism in each direction. Let $i : * \to I$ be the inclusion at 0. Show that $\mathcal{F} = RLP(\{i\})$.
2. Show that $\mathcal{W}$ verifies 2-out-of-3, and that $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ are stable under retracts.
3. a) Show that every functor F of $\mathcal{C} \cap \mathcal{W}$ has a left inverse G which is also a quasi-inverse and such that the natural transformation $FG \simeq \mathrm{id}$ is equal to the identity on the image of F.
   b) Deduce that $\mathcal{F} \subset RLP(\mathcal{C} \cap \mathcal{W})$.
4. Show that $\mathcal{C} \subset LLP(\mathcal{F} \cap \mathcal{W})$.
5. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Denote $\mathrm{Path}(F) := \mathcal{A} \times_{\mathcal{B}} \mathcal{B}^I$ where the map $\mathcal{B}^I \to \mathcal{B}$ is the source map, and $\mathrm{Cyl}(F) := \mathcal{A} \times (I \coprod_{\mathcal{A}} \mathcal{B})$. Show that F factors through $\mathrm{Path}(F)$ and $\mathrm{Cyl}(F)$; deduce that $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model structure on Cat.
6. What are the fibrant objects, the cofibrant objects ?

Exercise 10 Model structures on vector spaces (*Adapted from Najib Idrissi*)

Let k be a field and denote $\mathrm{Vect}(k)$ the category of vector space on k. We will use that every vector space, even the infinite-dimensional ones, has a basis (aka the axiom of choice).

1. Let there be a commutative square (without a dotted arrow for now)

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "i"'] & V \arrow[d, "p"] \\ F \arrow[r, "v"] \arrow[ur, dotted] & W \end{tikzcd}$$

   a) Show that u factors through i if and only if $\ker i \subset \ker u$.
   b) Show that v factors through p if and only if $\operatorname{im} v \subset \operatorname{im} p$.
   c) Show that there exists a dotted arrow as above if and only if both conditions are met.

2. a) Show that $i \perp p$ if and only at least one of i, p is surjective and at least one is injective.
   b) Deduce what are the possibilities for $LLP(\mathcal{W})$, when $\mathcal{W}$ is any class of arrows.

3. Suppose $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model structure on $\mathrm{Vect}(k)$.
   a) Show that $\mathcal{W} = RLP(\mathcal{F}) \circ LLP(\mathcal{C})$. Using the above, what are the possibilities for $\mathcal{W}$, $\mathcal{C}$, $\mathcal{F}$, $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{F} \cap \mathcal{W}$?
   b) Using that a model structure is fully determined by the data of $\mathcal{W}$ and $\mathcal{F}$, make a list of all the model structures on $\mathrm{Vect}(k)$ (be careful that you are writing model structures).

Exercise 11 Properness

A model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on $\mathcal{A}$ is said to be *left proper* if weak equivalences are stable under cobase change of cofibrations, and *right proper* if weak equivalences are stable under base change of fibrations.

1. In the classical model structures you know, which ones are right proper? left proper?
2. Show that if every object is fibrant, then $\mathcal{A}$ is right proper (resp. cofibrant, left proper).
3. Suppose that weak equivalences in $\mathcal{A}$ are stable under base change of fibrations with target (and thus source) a fibrant object. Show that $\mathcal{A}$ is right proper.