Compactifications and 6-functor formalism

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Following Lecture IV of [Sch24], the algebraic geometry drawn from various tags of the [Sta23], and some notes of Marc Hoyois' talk at the YTM 2023 in Lausanne. Also Marco Volpe's [Vol23].

Original material is limited to whatever errors have slipped past.

Contents

1 Compactifications in topology and algebra

Recall the following theorem, which we will attribute to Stone-Čech (the second assertion is actually due to Tychonoff in earlier work):

Theorem 1.1 — Stone-Čech compactification. Let X be a topological space. Then, there exists a continuous map $i : X \to \beta X$ which is initial among maps $X \to K$ such that K is compact Hausdorff. Moreover, if X is a Tychonoff space, i is an homeomorphism onto its image.

Tychonoff spaces, also known as $T_{\frac{3}{2}}$-spaces, encompass all locally compact Hausdorff spaces. In particular, for a map $f : X \to Y$ of locally compact Hausdorff spaces, one can always consider a relative compactification $X \to \overline{X} \to Y$ where $\overline{X}$ fits into the following cartesian square:

$$\begin{array}{ccc} \overline{X} & \longrightarrow & \beta X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \beta Y \end{array}$$

Definition 1.2 A map $p : A \to B$ of locally compact Hausdorff spaces is proper if the preimage of a compact subset is still compact. Equivalently, if p is closed and $p^{-1}(b)$ is compact for every $b \in B$.

Lemma 1.3 The map $i : X \to \overline{X}$ is an open immersion of topological spaces and the map $p : \overline{X} \to Y$ is proper.

Proof. Every continuous map between compact Hausdorff spaces is proper, and this notion is clearly closed under pullback.

An open immersion of topological spaces is nothing more than a homeomorphism onto its image. The image of i is the subspace $\{(\beta(x), f(x)) \mid x \in X\}$ of $\overline{X}$ where $\beta : X \to \beta X$, which is clearly homeomorphic to X since β is an open immersion. □

Let us now turn to a different setting: fix S a quasi-compact quasi-separated scheme.

Definition 1.4 A morphism of schemes $X \to S$ is said to be *separated* if the diagonal immersion $X \to X \times_S X$ is closed.

A morphism of schemes $X \to S$ is said to be of *finite type* if it is quasi-compact and locally of finite type in the following sense: for every $x \in X$, there is an affine neighborhood A of x and an affine neighborhood R of $f(\mathrm{Spec}(A))$ such that the map $R \to A$ induces a surjection $R[X_1, ..., X_n] \to A$ of R-algebra for some $n \in \mathbb{N}$.

Recall also the following two definitions:

Definition 1.5 An *open immersion* $X \to S$ is a morphism of schemes which is an open immersion of topological spaces and induces an equivalence $f^{-1}\mathcal{O}_S \cong \mathcal{O}_X$.

A proper map $X \to S$ is a morphism of schemes is a map which is separated, of finite type and universally closed in the following sense: for every map $S' \to S$, the induced $X \times_S S' \to S'$ is closed.

Theorem 1.6 — Nagata compactification. Let S a quasi-compact quasi-separated scheme. A morphism $X \to S$ is separated of finite type if and only if there exists a factorization

$$X \xrightarrow{i} \overline{X} \xrightarrow{p} S$$

with i an open immersion and p proper.

In the topological case, there was an initial such factorization but this is no longer the case. However, the category of such factorizations is cofiltered (given by taking adequate pullbacks) so it is in particular contractible.

2 Compactifications and the six operations

2.1 Open immersions and $f_!$

Let us now return to our favorite examples of six functor formalisms: $\mathrm{Shv}(X; \mathcal{D})$ for a locally compact Hausdorff space X and a presentable stable ∞-category $\mathcal{D}$ (which could be Sp, $D(\mathbb{Z})$, $D(R)$ for any ring spectra R, and maybe other weirder things).

A continuous map $f : X \to Y$ induces by definition a functor $f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X)$ between posets. Passing to presheaves, we get an adjunction pullback-pushforward:

$$f^* : \mathrm{Shv}(Y; D(\mathbb{Z})) \rightleftarrows \mathrm{Shv}(X; D(\mathbb{Z})) : f_*$$

where $f_*\mathcal{F}$ is the sheaf $V \mapsto \mathcal{F}(f^{-1}(V))$ and $f^*\mathcal{G}$ is the sheafification of $U \mapsto \mathrm{colim}_{f^{-1}(V) \supset U} \mathcal{G}(V)$, i.e. the left Kan extension of $\mathcal{G}$ along f^{-1}.

Now suppose f is an open immersion. Thus, in particular f is open so there is a functor $f : \mathrm{Open}(X) \to \mathrm{Open}(Y)$. Since f is an homeomorphism on its image, this functor is fully-faithful, and left adjoint to f^{-1} since $V \subset f(U) \Leftrightarrow f^{-1}(V) \subset U$. This adjunction descends at the level of $\mathcal{D}$-sheaves and realizes f^* as the functor which performs restriction to opens of X.

Lemma 2.1 Let $f : X \to Y$ be an open immersion. Then, f^* has a left adjoint $f_!$ given by the "extension by zero", which is the sheafification

$$f_!\mathcal{F}(V) := \begin{cases} \mathcal{F}(f^{-1}(V)) & \text{if } V \subset f(X) \\ 0 & \text{otherwise.} \end{cases}$$

of the above presheaf.

Proof. For the first part, it is easy to check that if $\mathcal{G}$ is a presheaf on X, then $f^*f_!\mathcal{G} \simeq \mathcal{G}$

naturally. Moreover, the induced map

$$\mathrm{Nat}(f_!\mathcal{F},\mathcal{G}) \xleftarrow{\simeq} \mathrm{Nat}(\mathcal{F}, f^*\mathcal{G})$$

is an equivalence. Indeed, when $\mathcal{F}$ is representable, which is enough for the adjunction on Sp-valued sheaves, this is clear by the Yoneda lemma, and using $\mathrm{Shv}(X;\mathrm{Sp})\otimes\mathcal{D}\simeq\mathrm{Shv}(X;\mathcal{D})$, this extends to general $\mathcal{D}$-valued sheaves. □

2.2 Proper maps and $f_!$

Since X is locally compact, there is an equivalence:

$$\bigcup_{V\subset K\subset U} V \xrightarrow{\simeq} U$$

where V ranges over open subsets of U contained in a compact K. In particular for any sheaf $\mathcal{F}$ on X, we have

$$\mathcal{F}(U) \xrightarrow{\simeq} \lim_{V\subset K\subset U} \mathcal{F}(V)$$

We can write this idea differently: it also holds that:

$$\operatorname*{colim}_{K\subset U} \mathcal{F}(U-K)\simeq 0$$

where K ranges over compact subsets of U. Let $\mathcal{F}_K(U) := \mathrm{fib}(\mathcal{F}(U)\to\mathcal{F}(U-K))$. Using that the target category $\mathcal{D}$ is stable, we have a fiber sequence.

$$\operatorname*{colim}_{K\subset U} \mathcal{F}_K(U) \xrightarrow{\simeq} \mathcal{F}(U) \to \operatorname*{colim}_{K\subset U} \mathcal{F}(U-K)\simeq 0$$

If $f: X\to Y$ is a continuous map, we have already seen a way to pushforward a sheaf $\mathcal{F}$ via the formula $f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$. But we could also try via the left hand side formula and write

$$f_!\mathcal{F}(V) := \operatorname*{colim}_{K\subset f^{-1}(V)} \mathcal{F}_K(f^{-1}(V))$$

There is a natural transformation $f_!\to f_*$, but since $f^{-1}(K)$ might not be compact in general, it is not clear that this map is an equivalence. What happens is that V might not be not be coverable by compacts $f(K)$ coming from the image of f.

Now suppose $f: X\to Y$ is proper, then we claim that the map $f_!\to f_*$ is actually an equivalence.

Lemma 2.2 Let $f: X\to Y$ be proper. Then, the natural transformation $f_!\to f_*$ is an equivalence.

Proof. Indeed, it is straightforward to check that properness implies that for every open V, the poset of compacts $f(K)\subset V$ is cofinal in the poset of compacts $K\subset V$, so that the above colimit coincides with the value at $f^{-1}(V)$ as wanted. □

Remark 2.3 — Volpe, 6.1 in [Vol23]. Another construction is the following: if $\mathcal{C}$ is presentable stable, then $\mathrm{Shv}(X;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ is equivalently given by $\mathrm{coShv}(X;\mathcal{C})$, which by Verdier duality is again equivalent to $\mathrm{Shv}(X;\mathcal{C})$, the adjunction $f^*\dashv f_*$ of $\mathcal{C}^{\mathrm{op}}$ induces an adjunction $f_!\dashv f^!$. One has to be careful, however, because $(-)^{\mathrm{op}}$ inverts the left and the right adjoint.

Here is a good moment to recall what we have seen in the previous section: for every morphism $X\to Y$, there exists a factorization $X\xrightarrow{i}\overline{X}\xrightarrow{p}Y$. where i is an open immersion and p proper. In particular, if we want our $(-)_!$ to be functorial, then it must be that $(p\circ i)_!$ coincides with the composite $p_!i_!$; in particular we have found a legitimate definition.

In the topological space setting, this factorization is canonical but this is not true for the schemes per our remark after Theorem 1.6. So, we are going to forget about this canonicity and work our way to showing that any two definition coincides.

Now suppose $f : X \to Y$ and $g : Y' \to Y$ are two maps such that g^* has a left adjoint $g_!$, one can form the base change square:

$$\begin{array}{ccc} X' := X \times_Y Y' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\quad g \quad} & Y \end{array}$$

By combining functoriality of $(-)^*$ and adjunctions, one gets a natural transformation

$$g_! f'_* \longrightarrow f_* g'_!$$

of functors $\mathrm{Shv}(X'; \mathcal{D}) \to \mathrm{Shv}(Y; \mathcal{D})$.

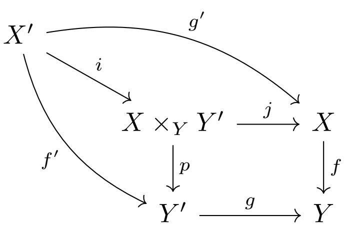
Theorem 2.4 Suppose $f : X \to Y$ is proper and $g : Y' \to Y$ is an open immersion, then the above natural transformation is an equivalence.

Corollary 2.5 Let there be any commutative square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

where f', f are proper and g', g open immersions. Then, there is an equivalence $g_! f'_* \simeq f_* g'_!$.

Proof. Form the following diagram:



Remark that it is automatic that i is an open immersion which is also proper. In particular, we get an equivalence $i_! \simeq i_*$ (note that in particular, the two possible definitions of $i_!$ must coincide). By the previous theorem, $g_! p_* \simeq f_* j_!$ and precomposing by the common value of $i_! \simeq i_*$ concludes by functoriality of $(-)_!$ on open immersions and of $(-)_*$ on proper maps. □

In particular, if $f : X \to Y$ is a composite of an open immersion and a proper map, then, there is a well-defined $f_!$ which does not depend from the choice of a composite.

Remark 2.6 We have ignored for simplicity the symmetric monoidal structure on each $\mathrm{Shv}(X, \mathcal{D})$ but all the above results can be improved to add projections formula stating further compatibilities.

3 General extensions of 6-functor formalisms

3.1 The extension theorem

Let $\mathcal{C}$ be a category and E a collection of morphisms of $\mathcal{C}$ which are stable by pullback, compositions and contain all isomorphisms. Recall that there is a symmetric monoidal category $\mathrm{Corr}(\mathcal{C}, E)$ whose objects are those of $\mathcal{C}$ and morphisms are spans whose covariant leg is in E; in particular, every morphism is the composite of a contravariant morphism $f^* : X \xleftarrow{f} Y \xrightarrow{\simeq} Y$ followed by a covariant morphism $g_! : X \xleftarrow{\simeq} X \xrightarrow{g} Y$ (where $g \in E$).

The holy grail of our quest is a lax-symmetric monoidal functor

$$\mathcal{D} : \mathcal{C}^{\mathrm{op}} \simeq \mathrm{Corr}(\mathcal{C}, E) \longrightarrow \mathbf{Cat}_\infty$$

such that each $D(X)$ is closed and the covariant and contravariant morphisms admit right adjoints (which are properties of such $\mathcal{D}$ which can be checked pointwise/arrow-wise. In the previous talk(s), we have seen how to produce the contravariant functoriality, i.e. a functor

$$\mathcal{D}_0 : \mathrm{Corr}(\mathcal{C}, \mathrm{Iso}) \longrightarrow \mathbf{Cat}_\infty$$

We now seek to explain one convenient way to extend this to more general E (which ties to the previous part).

Definition 3.1 A *compactification setting* (I, P) of E is the datum of two classes of arrows I, P stable under pullback, composites and containing all equivalences such that:

(i) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is such that $g \circ f, f \in I$ then $g \in I$ and the same condition for P.

(ii) Any $f \in I \cap P$ is n-truncated for some $n \in \mathbb{N}$.

(iii) Any map $f \in E$ admits a factorization $\overline{f} j$ with $j \in I$ and $\overline{f} \in P$.

We will refer to maps in P as proper and maps in I as open immersions.

Recall that a map $f : X \to Y$ is n-truncated if for every $g : Z \to Y$ the space of lifts $h : X \to Z$ such that $g \circ h \simeq f$ is n-truncated, i.e. has vanishing homotopy groups of order $> n$.

Theorem 3.2 — Mann, Liu-Zhang. Let (I, P) be a compactification setting for E and let $\mathcal{D}_0 : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ be a 3-functor formalism. Suppose the following holds:

(i) For every open immersion f, the functor f^* admits a left adjoint $f_!$ satisfying base change and the projection formula.

(ii) For every proper map p, the functor f^* admits a right adjoint f_* satisfying base change and the projection formula.

(iii) For any cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

where $f \in P$ and $g \in I$, the natural transformation $g_! f'_* \xrightarrow{\simeq} f_* g'_!$ is an equivalence.

Then, there exists a canonical extension $\mathcal{D} : \mathrm{Corr}(\mathcal{C}, E) \to \mathbf{Cat}_\infty$ of $\mathcal{D}_0$ such that the contravariant functoriality $(-)_!$ is given by $f_!$ for open immersions and f_* for proper maps.

Let us mention right away the following lemma, which is necessary for the above to be sensible, and explains (ii) of the definition of a compactification setting:

Lemma 3.3 In the context of the theorem, let f be a proper open immersion. Then $f_! \simeq f_*$.

Proof. By (ii), we know that such a f is n-truncated for some $n \in \mathbb{N}$. If $n = -2$, f is an isomorphism which concludes; we can thus work by induction.

Consider the diagonal $\Delta : X \to X \times_Y X$, it is $(n-1)$-truncated and moreover, denoting $g : X \times_Y X \to X$ the induced morphism, we have that $g \in I \cap P$ by pullback stability as well as $\mathrm{id}_X = g \circ \Delta$ by hypothesis, so $\Delta \in I \cap P$ is a prime target for our induction hypothesis.

Now, the result follow from formal manipulations: we have $f_! g_* \simeq f_* g_!$ by condition (iii) and $g \circ \Delta \simeq \mathrm{id}$ so precomposing by $\Delta_! \simeq \Delta_*$ concludes. □

3.2 A few words about the proof

We try to explain the sketch of proof given in [Sch24]. Let X be a bisimplicial set, then pulling back along the diagonal $\delta : \Delta \to \Delta \times \Delta$ induces a simplicial set $\delta^* X$. Differently written, $\delta^* X$ is the simplicial set given by the universal property:

$$\mathrm{Hom}(\Delta^n, \delta^* X) \simeq \mathrm{Hom}(\Delta^n \times \Delta^n, X)$$

We can also try the following thing:

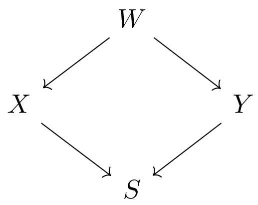
$$\mathrm{Hom}(\Delta^n, \delta_+^* X) \simeq \mathrm{Hom}(\Delta_+^{(n,n)}, X)$$

where $\Delta_+^{(n,n)}$ is the sub-bisimplicial set where we only keep vertices (i, j) when $i \geq j$. The inclusion $\Delta_+^{(n,n)} \to \Delta^n \times \Delta^n$ induces a map $\delta^* X \to \delta_+^* X$ which "deletes the bottom half of the square".

Now remark that $\mathrm{Corr}(\mathcal{C}, E)$ can be defined in the following way: first form the bisimplicial set $K(\mathcal{C}, E)$ whose (n, m)-simplices are maps

$$(\Delta^n)^{\mathrm{op}} \times \Delta^m \to \mathcal{C}$$

which send covariant maps to E and all possible squares to cartesian squares. Then $\delta_+^* K(\mathcal{C}, E)$ is the usual $\mathrm{Corr}(\mathcal{C}, E)$ but there is also a $\delta^* K(\mathcal{C}, E)$ which has edges



where the square is required to be cartesian. In general, $\delta^* K(\mathcal{C}, E)$ is not an ∞-category, as composing is particularly difficult (unless say $\mathcal{C}$ is stable and E is closed under pushouts as well).

The surprising fact is as follows:

Theorem 3.4 — Liu-Zheng. The map $\delta^* K(\mathcal{C}, E) \to \mathrm{Corr}(\mathcal{C}, E)$ is a categorical equivalence. In particular,

$$\mathrm{Fun}(\mathrm{Corr}(\mathcal{C}, E), \mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathrm{Fun}(\delta^* K(\mathcal{C}, E), \mathbf{Cat}_\infty)$$

is an equivalence.

This means we are reduced to providing maps $\delta^* K(\mathcal{C}, E) \to \mathbf{Cat}_\infty$ of simplicial sets.

Now, we can play this game again: assume we are in the setting of the theorem. Then, we can define a tri-simplicial set $K(\mathcal{C}, I, P)$ whose (n, m, k)-simplices are

$$(\Delta^n)^{\mathrm{op}} \times \Delta^m \times \Delta^k \to \mathcal{C}$$

which send the second coordinate to I, the third to P and so that every possible square or cube is cartesian. If δ also denotes the diagonal $\Delta \to \Delta \times \Delta \times \Delta$, then we have a map

$$\delta^* K(\mathcal{C}, I, P) \to \delta^* K(\mathcal{C}, E)$$

of simplicial sets.

Theorem 3.5 — Liu-Zheng. The map $\delta^* K(\mathcal{C}, I, P) \to \delta^* K(\mathcal{C}, E)$ is a categorical equivalence.

References

[Sch24] Peter Scholze. Six-functor formalisms. `https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf`, 2024.

[Sta23] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2023.

[Vol23] Marco Volpe. The six operations in topology. *Preprint*, 2023.