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# Méthodes de traces pour $\infty$-catégories stables

## Trace methodes for stable $\infty$-categories

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THÈSE DE DOCTORAT
présentée par

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pour l'obtention du grade de
DOCTEUR EN MATHEMATIQUES

soutenue le 2 juillet 2025 devant le jury d'examen composé de :

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# Trace methods for stable $\infty$-categories

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June 25, 2025

**Abstract**

**Résumé:** On reformule les idées de méthodes de trace pour comprendre la K-théorie algébrique à travers les outils de la théorie de l'homotopie moderne. En particulier, on étend la caractérisation de THH en tant que stabilisation de la K-théorie à toutes les catégories stables et les bimodules sur celles-ci. Plus généralement, on calcule toute la tour de Taylor-Goodwillie de la K-théorie lacée, notre extension de la K-théorie des endomorphismes paramétrisés. Pour ce faire, on fournit des propriétés universelles pour la K-théorie lacée et THH.

Dans un second temps, on étudie la convergence de la tour de Taylor-Goodwillie de la K-théorie lacée et de THH lacée. Cela amène à un résultat comparant la K-théorie lacée à l'homologie cyclique topologique lacée, expliquant la structure locale de la K-théorie algébrique. Pour cela, on introduit les structures coeurs sur les catégories stables, généralisant les structures de poids de Bondarko, et on produit des théorèmes de résolutions pour la K-théorie et THH. Le point d'orgue est une généralization du cas extension de carré nulle scindée du théorème de Dundas-Goodwillie-McCarthy pour les extensions nilpotentes qui s'applique désormais à des catégories stables qui ne sont pas nécessairement l'envelope stable d'une catégorie additive.

**Abstract:** We reframe the ideas of trace methods to understand algebraic K-theory through modern homotopy theory. In particular, we extend the characterization of THH as stable K-theory for all stable categories and all bimodules, and more generally, compute the whole Taylor tower of laced K-theory, our extension of K-theory of parameterized endomorphisms. In the process, we characterize both laced K-theory and THH with coefficients by universal properties.

In a second time, we study the convergence of the Taylor tower for laced K-theory and laced THH, which gives rise to comparison results with topologic cyclic homology, and the so-called local structure of algebraic K-theory. For this, we introduce heart structures on stable categories generalizing the weight structures of Bondarko and leverage them to extend resolution theorems for K-theory and THH. The upshot is that we are able to generalize the split square-zero case of the Dundas-Goodwillie-McCarthy theorem to stable categories that are not generated by an additive subcategory.

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# 1 Introduction

## 1.1 What are trace methods?

Let $M$ be a square matrix, then $\det(\mathrm{id} + tM)$ is a polynomial in the variable $t$ whose constant term is 1. Let us denote for a moment $P_1 \det(M)$ the linear coefficient: it is a classical exercise of linear algebra to determine it, and let us run quickly through what we believe is the most elegant method. First, remark that for any (non-necessarily square!) matrices $A, B$, we have $\det(\mathrm{id} + tAB) = \det(\mathrm{id} + tBA)$, a formula known as the Weinstein–Aronszajn identity; this long-winded name hides its simplicity, as it can be obtained from considering $\det(A + ABA)$ and using in short succession, the multiplicative property of det, the commutativity of the base ring and the polynomial nature of the determinant.

Identifying the linear coefficients of the two polynomials, it must also hold that $P_1 \det(AB) = P_1 \det(BA)$. Moreover, it is also true that $P_1 \det(M)$ is linear in $M$; but then this fully-determines our application up to a multiplicative constant. Indeed, $P_1 \det$ is determined by its values on $(E_{ij})$, the canonical basis of square matrices, but

$$E_{ij}E_{kl} = \delta_{jk}E_{il}$$

where $\delta_{jk}$ is the Kronecker symbol, which vanishes unless $j = k$ and equals one otherwise. In particular, the cyclic invariance implies that $\delta_{jk}P_1 \det(E_{il}) = \delta_{li}P_1 \det(E_{kj})$, from which we deduce that if $i \neq l$, $P_1 \det(E_{il}) = 0$ and on diagonal matrices, $P_1 \det(E_{ii}) = P_1 \det(E_{jj})$. In our case, it is not too hard to determine the constant, which is 1, so that the mysterious $P_1 \det$ is revealed to be none other than the trace tr.

More can be said if the base field is of characteristic zero: then there exists a matrix $\exp(M)$ satisfying to the following equation $\det(\exp(M)) = \exp(\mathrm{tr}(M))$. Moreover, writing $L := \ln(1+tM)$ for the value of infinite series of general term $\frac{(-1)^{n+1}t^nM^n}{n}$ (this converges by forcing $t$ to be small), we get a matrix $L$ such that $\exp(L) = 1 + tM$ and we deduce:

$$\ln\det(\mathrm{id} + tM) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{\mathrm{tr}(M^n)}{n} t^n \tag{1}$$

Passing again to the exponential, and developing the series that defines it, we get an expression of $\det(\mathrm{id} + tM)$ for sufficiently small values of $t$. We claim that this is precisely the essence of trace methods:

> **Slogan — Trace methods.** Knowledge of traces of powers of a matrix $M$ is sufficient to recover the whole determinant of $\mathrm{id} + tM$ if $t$ is small enough.

The goal of this thesis is, broadly, to show that this idea is so sturdy that it will survive quite a leap in abstraction. Namely, we will show that it can be made sense of the (logarithm of the) *determinant of a bimodule $M$ over a stable $\infty$-category $\mathcal{C}$* relative to the determinant of the base, an object we denote $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, M)$, and that under good hypotheses on $\mathcal{C}$ and $M$, it can be recovered via knowledge of iterated traces $\mathrm{THH}(\mathcal{C}, M^n)$. Moreover, we will give a meaning to the "Taylor tower of $\mathrm{K}^{\mathrm{cyc}}$" and show in full generality that it is given by a formula which is essentially a generalization of Equation 1, see Corollary 5.34 and Theorem 6.18.

Let $R$ a ring and $M$ a $R$-bimodule, then we let $\mathrm{K}_0(\mathrm{End}(R, M))$ be the abelian group generated by symbols $[N, f]$ for every finite-type projective left $R$-module $N$ and every $R$-linear endomorphism $f : N \to M \otimes_R N$, under the following two relations:

- $[N, f + g] = [N, f] + [N, g]$
- For every commutative diagram with rows short-exact sequences of left $R$-modules

$$\begin{tikzcd} N \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d, "g"] & Q \arrow[d, "h"] \\ M \otimes_R N \arrow[r] & M \otimes_R P \arrow[r] & M \otimes_R Q \end{tikzcd}$$

  the relation $[P, g] = [N, f] + [Q, h]$

Suppose $M = R$ for a moment; then, since $N$ is projective of finite type, we have

$$\mathrm{Hom}_R(N, N) \simeq \mathrm{Hom}_R(N, R) \otimes_R \mathrm{Hom}_R(R, N)$$

and the evaluation map provides a well-defined morphism $\mathrm{End}_R(N) \to R$. By post-composing by $R \to R/[R, R]$, it can be shown that the above morphism descends to a morphism to the $\mathrm{K}_0$-group previously defined; in fact this works with general $M$ by a slight adaptation

$$\mathrm{K}_0(\mathrm{End}(R, M)) \longrightarrow R/[R, R] \otimes_R M =: \mathrm{HH}_0(R, M)$$

which, in the case where $M = R$, sends $f : N \to N$ to $\mathrm{tr}(f)$, the trace of $f$. This categorification of the trace is called the *Hattori-Stallings trace*.

The abelian group $\mathrm{K}_0(\mathrm{End}(R, M))$ appearing on the left of the Hattori-Stallings trace happens to be a K-group, as we have hinted by the notation — an instance of a general construction first introduced by Grothendieck. Let $\mathcal{E}$ be an exact category, that is an additive category with a specification of short-exact sequences which satisfy some closure properties, then the direct sum endows the core groupoid of $\mathcal{E}$ with the structure of a commutative monoid, up to isomorphisms. Modding out by the latter, one can further enforce that every short-exact sequence splits by adding the corresponding relations and that every element has an inverse, making our commutative monoid into an abelian group. The resulting object is usually denoted $\mathrm{K}_0(\mathcal{E})$ and the above furnishes a universal property for it.

If we let $\mathrm{End}(R, M)$ denote the category of $R$-linear endomorphisms $f : N \to M \otimes_R N$ for finitely generated projective $N$, with arrows given by the commutative squares, then we have an additive category and we can endow it with an exact structure whose exact sequences are exactly given as in condition (ii) previously. It is clear that the two constructions of $\mathrm{K}_0(\mathrm{End}(R, M))$ we have given do agree.

As the notation suggests, there is more than one K-group and $\mathrm{K}_0(\mathcal{E})$ is but the zeroth homotopy group of a grouplike $\mathbb{E}_\infty$-space[1] denoted $\mathrm{K}(\mathcal{E})$ following Quillen in [Qui73]. Note that grouplike $\mathbb{E}_\infty$-space are equivalently described as a connective spectra without change to the homotopy groups, a perspective we will adopt now. The connective K-theory spectrum, through its homotopy groups, captures information critical to many solved and unsolved problems – let us cite in the latter category the Kummer-Vandiver conjecture. Unfortunately, its computation is extremely hard, and as the last example proves, even the complete K-theory of the integer is partly conjectural.

On the other side of the Hattori-Stallings trace, the group is usually known as $\mathrm{HH}_0(R, M)$; in fact, it is the zeroeth group of a cohomology theory whose representing spectrum we denote $\mathrm{HH}(R, M)$, the *Hochschild homology* of $R$ with coefficients in $M$, which is such that the Hattori-Stallings trace identifies as the map on $\pi_0$ of a map of spectra:

$$\mathrm{K}(\mathrm{End}(R, M)) \longrightarrow \mathrm{HH}(R, M)$$

This is quite a rough map: it loses much of the K-theory information. After all, the trace is certainly not enough to recover a determinant, and so it goes for the categorified story. The first key insight of trace methods is that Hochschild homology has extra structure: $\mathrm{HH}(R, M^n)$ has a cyclic action of order $n$ and when taking the unit $M = R$, all of those actions assemble into a $S^1$-action.

Endowing K-theory with the trivial $S^1$-action, the above map factors through $\mathrm{HH}(\mathrm{End}(R, M))$ with coefficients in itself via a $S^1$-equivariant map. From this, we get the *Goodwillie-Jones* trace:

$$\mathrm{K}(\mathrm{End}(R, M)) \longrightarrow \mathrm{HC}^-(\mathrm{End}(R, M))$$

where $\mathrm{HC}^-$, the negative cyclic homology, is the homotopy fixed points under the $S^1$-action of $\mathrm{HH}(\mathrm{End}(R, M)$. A theorem of Goodwillie says that rationally, a slight variation on the above map (replacing $\mathrm{End}(R, M)$ by the category of projective finite type $R \oplus M$-modules, where $R \oplus M$ is the square-zero extension of $R$ by the bimodule $M$) is an equivalence, up to modding out by $\mathrm{K}(R)$ on the left and $\mathrm{HC}(R)$ on the right.

Now is a good moment to say what already permeates the above discussion: we have a "homotopy-pest problem". Not only all of the above facts can be extended to a more general, homotopy-theoretic setup, but actually, some of the above facts only hold in a suitable homotopic sense. Instead of harrowingly and endlessly trying to chase the homotopy out, we have decided instead to make peace with them, and to integrate them within the essence of our questions — we will now pass to the framework of $\infty$-categories initially developed by Joyal and then Lurie, and to cement that this is a natural choice, we will omit the $\infty$ in front of every higher category, as well as the homotopy in front of our limits, colimits and functors (and save some rare exceptions, in front of everything). Hopefully once this thesis is complete, the skeptical reader will be convinced

[1] Here and unless otherwise specified, we use the word *space* to talk about homotopy types of spaces, sometimes called *anima*.

that this cohabitation was the only sensible choice and that after all, these homotopies are not so pesky as one would have been led to believe.

In [Bar16], Barwick developed a definition of K-theory adapted to exact categories, refining Quillen for exact 1-categories. For stable categories, in some sense the "maximally exact" categories, which are equipped with a t-structure, he also proved in [Bar15] that the canonical map $\mathcal{C}^\heartsuit \to \mathcal{C}$ from the abelian 1-category which the heart of the t-structure on $\mathcal{C}$ to $\mathcal{C}$ itself, is sent to an equivalence by K-theory. Taking $\mathcal{C} = D^b(R)$ to be the derived bounded category of a ring $R$, we get that K-theory of stable categories recovers K-theory of ordinary rings.

For such categories, and slightly more generally, derived bounded categories of abelian categories, this also fits within the context of Gillet-Waldhausen-style theorems. Remark that the full subcategory of stable categories $\mathbf{Cat}^{\mathrm{Ex}}$ is a reflexive subcategory of $\mathbf{Exact}_\infty$, the category of exact categories and exact functors. The left adjoint, Stab, comes with a unit map $\mathcal{E} \to \mathrm{Stab}(\mathcal{E})$, which is such that

$$\mathrm{K}(\mathcal{E}) \longrightarrow \mathrm{K}(\mathrm{Stab}(\mathcal{E}))$$

is an equivalence. In this generality, this result is a consequence of work of the author and Christoph Winges in [Sau23b, SW25], but many cases have been known before, notably by work of Waldhausen, Gillet and Thomason. It in particular applies to exact 1-categories, and compares their K-theory to that of a stable category which is never discrete. In particular, the set-up of stable categories recovers entirely Quillen's definition of K-theory and it happens to be categorically nicer, so we will settle there for most of this thesis.

The general philosophy that one should garner from those statements is that K-theory is a profoundly homotopic object and that the perspective of viewing usual objects of algebra as part of the homotopic world is quite natural in the K-theoretic context. This is not quite the case of Hochschild homology: to bridge this gap, Waldhausen imagined that there should be a invariant which behaves similarly to $\mathrm{HH}(R, M)$ but understands the topology[2] better. In particular, for this topological version of Hochschild homology, the $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}$ would play the role of the ring of integers $\mathbb{Z}$.

The first construction of this *topological Hochschild homology*, denoted THH, is due to Bökstedt who also made the first important computations. For a connective $\mathbb{E}_\infty$-ring spectrum $R$, it was shown that there is a map

$$\mathrm{K}(R) \longrightarrow \mathrm{THH}(R)$$

which is usually called the *Dennis trace map.* Setting up the theory correctly, one sees that this is a map of spectra. Again, there is a $S^1$-action at the target, and one can extract a new spectrum $\mathrm{TC}^-(R)$, negative topological cyclic homology; here a twist to the story appears: the $S^1$-spectrum $\mathrm{THH}(R)$ has even more structure. Indeed, in the homotopy world lives the Tate construction $(-)^{\mathrm{tC}_p}$, an endofunctor of $S^1$-spectra and there are $S^1$-equivariant maps

$$\phi_p : \mathrm{THH}(R) \longrightarrow \mathrm{THH}(R)^{\mathrm{tC}_p}$$

Such a structure cannot be seen on HH: one way to understand this is to remark that due to blue-shifting phenomena, the Tate construction vanishes rationally and HH is only equipped to understand K-theory rationally. The $\phi_p$ are functorial and endow $\mathrm{THH}(R)$ with a *cyclotomic structure* and, though it is slightly more complicated, though of a similar flavor, than taking homotopy fixed points, one can extract another spectrum $\mathrm{TC}(R)$, topological cyclic homology, out of this structure so that the Dennis trace factors through as a map

$$\mathrm{K}(R) \longrightarrow \mathrm{TC}(R)$$

which is called the *cyclotomic trace map.* The upshot of this homotopy-upgrade-program is a result of Dundas-Goodwillie-McCarthy [DGM13], stating that the following square

$$\begin{tikzcd} \mathrm{K}(R \oplus M) \arrow[r] \arrow[d] & \mathrm{TC}(R \oplus M) \arrow[d] \\ \mathrm{K}(R) \arrow[r] & \mathrm{TC}(R) \end{tikzcd}$$

[2]Actually, it would probably fit better to say the "homotopy": although the two are tightly interlinked, it is the former that is more at the heart of this refinement but given the interests of Waldhausen and how early this name was coined, the resulting invariant was called THH and not HHH.

is cartesian. In other words, Goodwillie's rational result can be upgraded to an integral statement when passing to topological versions of all the invariants on the Hochschild homology side.

This result leads a treasure-trove of new computations in K-theory, featuring prominently in [AKN24] which computes $\mathrm{K}(\mathbb{Z}/p^n)$ or in a more conceptual way, in [BHLS23], which disproves an old conjecture of chromatic homotopy theory, the telescope conjecture, by understanding sufficiently finely the K-theory of a specific ring spectrum through a good understanding of its topological cyclic homology.

The Dundas-Goodwillie-McCarthy theorem itself sits atop a heap of arduous and often technical arguments, the setting of which having grown out of fashion thanks to the recent development of higher category theory and homotopy theory – making its proof hard to access for newcomers. Also, where as the early parts of our exposition were categorical, sometimes even enjoying universal properties, the historical approach to trace methods relies on connective ring spectra so that simplicial constructions and subsequent comparisons can be performed between related K-theoretic objects.

The goal of this text is twofold: we want to generalize the many statements of trace methods to more general objects but also, and maybe even more importantly, we want to shift the brunt of the proof away from the "*innards of K-theory*". By this, we mean that our proof strategies will be leveraging the fact that in a suitably general context, all the objects of the story of trace enjoy universal properties. The paradigm shift can be resumed as follows: we will prove results about the properties of K-theory to understand it better, rather than on the object itself. Incidentally, this means our story is much more model-independent.

We claim, and hope to demonstrate in what follows, that this makes the arguments simpler, at least if one is only interested in recovering the results that are already known. The "deal with the devil" we have made is that our proofs will therefore rely on the higher categorical world and cannot be used without this context; this is little to give up in our minds, especially given the growing interest in so-called higher algebra.

## 1.2 Structure of the thesis

Before we explain the content of the thesis, let us draw the following diagram explaining the structure of the manuscript:

$$\begin{tikzcd}
 & \S3 \arrow[d] & & & & \\
{[HNS24]} \arrow[ur, dash, dotted] \arrow[r, dash, dotted] \arrow[dr, dash, dotted] & \S4 \arrow[d] & & {[HNS25]} \arrow[ddl, bend left] \arrow[dddl, bend left] & & \\
 & \S5 \arrow[r] & \S6 \arrow[d] & & & \\
 & & \S7 \arrow[d] & & {[Sau23b]} \arrow[d, dash, dotted] & {[SW25]} \arrow[dl, dash, dotted] \\
 & & \S8 \arrow[dr] & \S9 \arrow[d] & \S10 \arrow[dl] & \\
 & & & \S11 & &
\end{tikzcd}$$

The section §2 is a short recollection of the preliminary notions that are used throughout the thesis. Plain lines indicate logical dependency whereas dotted lines serve to indicate which sections of thesis are contained in which preprints and/or publications.

Though it is still in project at the time of writing, we plan to include §9, §11 (and whatever part §8 which does not make it into [HNS25]).

Vertical columns serve to indicate sections that go together: §3-5 are about setting up laced K-theory and its linearization. §6-8 seek to investigate the link between higher derivatives of laced K-theory and extra structure on its first derivative; they also investigate the related question for laced topological Hochschild homology as a functor to genuine cyclotomic spectra. §9 develops an abstract criterion to check that a functor converges partially to the limit of its Taylor tower. §10 develops the theory of heart structures and proves resolutions theorems for laced K-theory and laced THH. Finally, §11 checks that heart structures can be used to produce a setting in which the criterion of §9 holds so that the knowledge of the Taylor towers acquired in §8 provides a concrete pullback square of spectra linking K-theory and topological cyclic homology.

Finally, let us point of that four Theorems are left unproven and at the time of writing, their proof is not yet available (they will be proven in [HNS25]): these are Theorems 7.18, 7.25, 7.28 and 8.4 — they impact only the proof of the main result, namely Theorem 11.4.

## 1.3 Survey of the results

Let us now come to the content of this work. The results listed in what follows, have been (or for some at the time of writing, will be) also recorded in the following articles [HNS24, HNS25, Sau23b, SW25], except for the result about convergence of Goodwillie-Taylor for which another article be written but too far in the future for us to give a reference.

The central idea we want to push is that trace methods is a *categorification* of the phenomena highlighted at the beginning of the introduction, relating determinants and traces. Hence, to be pursued the most successfully and the most efficiently, it is important to build the correct categories to host our invariants and this is what is usually the hardest part of the work.

The first element we want to categorify is the operation $\mathrm{id} + tM$ for a small $t$. For rings, there is a well-known construction that plays this role: given a $R$-bimodule $M$, one can endow the abelian group $R \oplus M$ with a ring structure such that the projection $R \oplus M \to R$ is a ring map and the square of two elements of $M$ is zero, hence the name of (split) square-zero extensions. Such split square-zero extensions have been thoroughly studied, and in the homotopical context which we care about, Lurie provides in [Lur17a] an abstract framework to produce square-zero extensions in a general presentable category $\mathcal{C}$.

Given $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$, the category $\mathrm{Bimod}(R)$ can be obtained by stabilizing $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})_{/R}$ and the canonical $\Omega^\infty : \mathrm{Bimod}(R) \to \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})_{/R}$ coincides with the usual split square zero extension of rings and the more general one defined in *loc. cit.* for ring spectra. In general, if $X \in \mathcal{C}$, Lurie write $\mathrm{T}_X\mathcal{C}$ for the stabilization of $\mathcal{C}_{/X}$ (or equivalently $\mathcal{C}_{X//X}$); such an association is functorial in $X$ and the functor unstraightens to a fibration denoted $\mathrm{T}\mathcal{C} \to \mathcal{C}$. Our first result is as follows, see Proposition 3.3 and Proposition 3.12:

**Proposition 1.1 — Harpaz-Nikolaus-S.** Let $\mathcal{C} \in \mathbf{Cat}^{\mathrm{Ex}}$. There is an equivalence of categories

$$\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Ind}(\mathcal{C}))$$

and the square-zero extension functor is given by $\mathrm{Lace}(\mathcal{C}, -)$, which fits in the following pullback square for $M : \mathcal{C} \to \mathrm{Ind}\,\mathcal{C}$:

$$\begin{tikzcd}
\mathrm{Lace}(\mathcal{C}, M) \arrow[r] \arrow[d] & \mathrm{Ind}(\mathcal{C})^{\Delta^1} \arrow[d] \\
\mathcal{C} \arrow[r, "{(j,M)}"] & \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C})
\end{tikzcd}$$

In particular, if $\mathcal{C} = \mathrm{Perf}(R)$ and $M$ is a $R$-bimodule, then we have

$$\mathrm{Lace}(\mathrm{Perf}(R), M \otimes_R -) \simeq \mathrm{End}(R, M)$$

We see from the above result that already at its beginning, our formalism achieved an interesting thing: it unified the two potential candidates for a square-zero extension that differed by a shift in the world of rings: $R \oplus M$ and $\mathrm{End}(R, M)$.

We call the category $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, bundled from the $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$ over $\mathbf{Cat}^{\mathrm{Ex}}$, the *tangent bundle of* $\mathbf{Cat}^{\mathrm{Ex}}$. We also coin a name for its objects: a pair $(\mathcal{C}, M)$ is called a *laced category*, so that $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is also the category of laced categories. The functors $\mathrm{Lace}(\mathcal{C}, -)$ can be upgraded to a global right adjoint functor $\mathrm{Lace} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$, whose left adjoint is the cotangent complex, explicitly given by $L : \mathcal{C} \mapsto (\mathcal{C}, \mathrm{map}_{\mathcal{C}})$.

Left Kan extending along the cotangent complex, or equivalently precomposing by Lace, we can lift any invariant of $\mathbf{Cat}^{\mathrm{Ex}}$ to a *laced invariant* $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$. Namely:

**Definition 1.2** We let *laced K-theory* be the composite functor

$$\mathrm{K}^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \xrightarrow{\mathrm{Lace}} \mathbf{Cat}^{\mathrm{Ex}} \xrightarrow{\mathrm{K}} \mathrm{Sp}$$

There is a notion of laced-additivity for invariants out of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, inspired by the additivity which gives $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ its universal property in [BGT13].

**Definition 1.3** Let $(\mathcal{C}, M)$ be a laced category. A *laced semi-orthogonal decomposition* of $(\mathcal{C}, M)$ is a pair of laced categories $((\mathcal{A}, N); (\mathcal{B}, P))$ with the following extra data and conditions:

(Underlying) The underlying pair of stable categories $(\mathcal{A}, \mathcal{B})$ is a semi-orthogonal decomposition of $\mathcal{C}$.

(Laced sub-categories) The inclusions refine to laced functors $(i, \alpha) : (\mathcal{A}, N) \to (\mathcal{C}, M)$ and $(j, \beta) : (\mathcal{B}, P) \to (\mathcal{C}, M)$ where $\alpha : N \simeq M \circ (i^{\mathrm{op}} \times i)$ and $\beta : P \simeq M \circ (j^{\mathrm{op}} \times j)$ are equivalences.

(Laced semi-orthogonality) For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $M(A, B) \simeq 0$.

Every such decomposition induces an exact sequence in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$

$$(\mathcal{A}, N) \longrightarrow (\mathcal{C}, M) \longrightarrow (\mathcal{B}, P)$$

where the first map is induced by the inclusion and the second by the adjoint to the inclusion. In particular, the adjoints to the inclusion are also laced functors so that the whole underlying additive sequence lifts to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$.

One can show that the adjunction $L \dashv \mathrm{Lace}$ preserves additive sequences, so that $\mathrm{K}^{\mathrm{lace}}$ is still additive. In fact, we have better by very little more work, see Theorem 4.10:

**Theorem 1.4 — Harpaz-Nikolaus-S.** The natural transformation $\Sigma^\infty_+ \iota\,\mathrm{Lace} \longrightarrow \mathrm{K}^{\mathrm{lace}}$ is initial among transformations $\Sigma^\infty_+ \iota\,\mathrm{Lace} \to F$ where $F$ is laced-additive.

Moreover, there is a lax-monoidal structure on $\mathrm{K}^{\mathrm{lace}}$ which makes it the initial lax-monoidal laced-additive functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$.

Here, the lax-monoidality is with respect to a symmetric monoidal structure on $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ which is straightforward to set-up, and compatible with that of $\mathbf{Cat}^{\mathrm{Ex}}$.

It follows from abstract considerations that there exists a functor $\mathrm{P}_1^{\mathrm{fbw}}\,\mathrm{K}^{\mathrm{lace}}$, called *stable K-theory*, and a natural transformation $\mathrm{K}^{\mathrm{lace}} \to \mathrm{P}_1^{\mathrm{fbw}}\,\mathrm{K}^{\mathrm{lace}}$ which is initial among natural transformation with source $\mathrm{K}^{\mathrm{lace}}$ and target a functor which is fiberwise-exact (see Lemma 5.26). Here fiberwise-exactness refers to the exactness of every $F(\mathcal{C}, -) : \mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ where $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is some functor to a stable category. Using the above theorem and the compatibility of the formation of fiberwise-exact approximations, one can show that $\mathrm{P}_1^{\mathrm{fbw}}\,\mathrm{K}^{\mathrm{lace}}$ also has the following universal property: the natural transformation it receives from $\Sigma^\infty_+ \iota\,\mathrm{Lace}$ is initial within those whose target is a laced-additive, fiberwise-exact functor.

We seek to understand more precisely the interaction between the additivity and the fiberwise-exactness condition. We note the following: there is a laced-category $(\mathcal{C}, M)^{([1],*)}$ whose underlying category is $\mathrm{Ar}(\mathcal{C})$ and whose bimodule is given by

$$M^{([1],*)}(f : X \to Y, g : X' \to Y') := M(Y, X')$$

This category admits a semi-orthogonal decomposition by $(\mathcal{C}, 0)$ and $(\mathcal{C}, M)$ so that fiberwise-reduced, additive invariant send the two maps $(\mathcal{C}, M)^{([1],*)} \to \mathcal{C}$ to an equivalence.

**Definition 1.5** We say that a functor $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is *trace-like* if it inverts the laced functors

$$d_0, d_1 : (\mathcal{C}, M)^{([1],*)} \longrightarrow \mathcal{C}$$

for every $(\mathcal{C}, M)$.

This is not quite how this notion is defined in Definition 5.15, but Proposition 5.17 guarantees it agrees with it.

In particular, we can interpret this as some kind of homotopy-invariance for a funny type of homotopy: namely a *trace homotopy*, i.e. a functor $H : (\mathcal{D}, N) \to (\mathcal{C}, M)^{([1],*)}$, can be thought as a homotopy between the two possible composite $(\mathcal{D}, N) \to (\mathcal{C}, M)$ and trace-like functors are those invariant under such trace homotopies.

By constructing the associated simplicial object $(\mathcal{C}, M)^{([n],*)}$, we can give a construction of the way to force a functor to be trace-like, see Theorem 5.21:

**Theorem 1.6 — Harpaz-Nikolaus-S.** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be any functor. Then, there exists an initial natural transformation with source $F$ whose target is a trace-like invariant, denoted $F \to \mathrm{cyc}(F)$ and its target $\mathrm{cyc}(F)$ satisfies:

$$\mathrm{cyc}(F)(\mathcal{C}, M) := \left| F\left( (\mathcal{C}, M)^{([\bullet],*)} \right) \right|$$

for every laced $(\mathcal{C}, M)$.

Putting $\iota\,\mathrm{Lace}$ into the above machine, we get a invariant usually known as *unstable topological Hochschild homology*, denoted uTHH. It follows that THH, by definition the fiberwise-exact approximation of $\Sigma^\infty_+ \mathrm{uTHH}$ — but using the previously-given formula for cyc, we will show that it coincides with the well-known construction of such a functor — is the initial trace-like, fiberwise-exact functor. Note that by the above considerations, we know those properties are implied by fiberwise-exact, laced-additivity so that THH receives an essentially unique natural transformation from $\mathrm{P}_1^{\mathrm{fbw}}\,\mathrm{K}^{\mathrm{lace}}$ under $\Sigma^\infty_+\,\iota\,\mathrm{Lace}$. Actually, it turns out that those properties also imply fiberwise-exact and laced-additivity, see Theorem 5.33 and Corollary 5.34 in the text:

**Theorem 1.7 — Harpaz-Nikolaus-S.** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a fiberwise-exact functor. Then, $F$ is laced-additive if and only if $F$ is trace-like.

In particular, the map $\mathrm{K}^{\mathrm{lace}} \longrightarrow \mathrm{THH}$ identifies its target as the fiberwise-exact approximation of its source, i.e. there is a canonical equivalence $\mathrm{P}_1^{\mathrm{fbw}}\,\mathrm{K}^{\mathrm{lace}} \simeq \mathrm{THH}$.

This last identification is precisely a generalization of the identification between stable K-theory and THH of [DM94]. Let us note two things about the proof. First, this proof has never quite "looked inside" both $\mathrm{K}^{\mathrm{lace}}$ and THH, except to check that the old construction had the wanted universal properties: the crux of it is the comparison of two properties, trace-like and additivity, which coincide under the further hypothesis of fiberwise-exactness. Second, our strategy is quite all-or-nothing: we can only compute the exact approximation of $\mathrm{K}^{\mathrm{lace}}(\mathcal{C}, -)$ because we can make $\mathcal{C}$ vary.

This is how far we will go into trace methods by staying in the world of laced categories. More precisely, although we will attempt to show how some ideas and computations can be made, it becomes increasingly tedious not to take the following change of point of view: we consider laced categories as graphs with one vertex with value $\mathcal{C}$ and one map from the vertex to itself, with value the $(\mathcal{C}, \mathcal{C})$-bimodule $M$.

To generalize, we consider cyclic graphs of arbitrary (finite) sizes, with arrows given by $(\mathcal{C}, \mathcal{D})$-bimodules where $\mathcal{C}, \mathcal{D}$ are the values of the adjacent vertices. These cyclic graphs have a number of natural operations: contractions and degeneracies which we can encode via a simplicial structure, rotation which we can encode via a cyclic structure using Connes' cyclic category $\Lambda$, and degree $k$ maps which we can encode via Goodwillie's epicyclic category $\Lambda^{\mathrm{epi}}$.

We have the following, which is the content of Lemma 7.2, Proposition 7.4 and Proposition 7.5:

**Proposition 1.8 — Harpaz-Nikolaus-S.** There is a commutative diagram whose squares are pullback and vertical arrows are cocartesian fibrations:

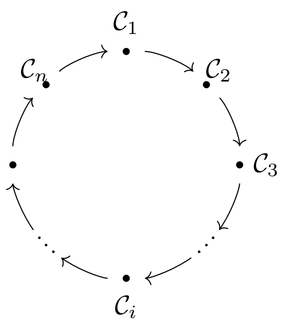
$$\begin{tikzcd}
\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \arrow[r] \arrow[d] & \mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}) \arrow[r] \arrow[d] & \mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}}) \arrow[r] \arrow[d] & \mathrm{THH}^{epi}(\mathbf{Cat}^{\mathrm{Ex}}) \arrow[d] \\
\{*\} \arrow[r, "{[0]}"] & \Delta^{\mathrm{op}} \arrow[r] & \Lambda^{\mathrm{op}} \arrow[r] & (\Lambda^{\mathrm{epi}})^{\mathrm{op}}
\end{tikzcd}$$

Over some $[n] \in \Delta$, the common fiber of the latter three cocartesian fibrations is the unstraightening of the functor whose value on object is given by:

$$\begin{aligned}
(\mathbf{Cat}^{\mathrm{Ex}})^{n+1} &\longrightarrow \mathbf{CAT} \\
(\mathcal{C}_0, ..., \mathcal{C}_n) &\longmapsto \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}_i^{\mathrm{op}} \otimes \mathcal{C}_{i+1}, \mathrm{Sp})
\end{aligned}$$

The names $\mathrm{THH}^*(\mathbf{Cat}^{\mathrm{Ex}})$ have been suggestively chosen to indicate that those categories are 2-categorical versions of THH (in a similar, though not precisely the same, way to Ponto-Shulman's THH of bicategories, see [PS13]). More precisely, as the unstraightening of a simplicial object, $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ is an oplax geometric realization whose simplicies are quite similar to a cyclic Bar construction; moreover, this simplicial object is in fact restricted from a cyclic, and in fact an epicyclic object, which entails to extra-structure.

Concretely, an object of $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ living over $[n] \in \Delta$ is the datum of a cyclic graph:




where each $\mathcal{C}_i$ is a stable category and each arrow corresponds to a point $M_i \in \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}_i^{\mathrm{op}} \otimes \mathcal{C}_{i+1}, \mathrm{Sp})$ or equivalently a continuous functor $M_i : \mathrm{Ind}(\mathcal{C}_{i+1}) \to \mathrm{Ind}(\mathcal{C}_i)$ (note the order reversal). In the fiber over $[n] \in \Delta$, maps are given by exact functors between the vertices and lax-commuting squares between each bimodule. To finish the description, let us also comment on the cocartesian edges: those living over a face compose the identified arrows whereas those over a degeneracy add bimodules equal to the identity in the graph.

The passage to $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$ add new cocartesian edges corresponding to rotations of the graph and further going to $\mathrm{THH}^{epi}(\mathbf{Cat}^{\mathrm{Ex}})$, one has access to $n$-fold covers of the circle as cocartesian edges, where $n \in \mathbb{N}$ is the degree of the underlying map of circles.

**Definition 1.9** A functor $F : \mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathcal{E}$ is *cyclic-invariant* if it inverts cocartesian arrows over $\Delta^{\mathrm{op}}$.

Similarly, a functor $F : \mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathcal{E}$ is *cyclic-invariant* if it inverts cocartesian arrows over $\Lambda^{\mathrm{op}}$.

Finally, a functor $F : \mathrm{THH}^{epi}(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathcal{E}$ is *cyclic-invariant* if it inverts cocartesian arrows over degree 1 maps, i.e. precisely those in the image of $\Lambda \to \Lambda^{\mathrm{epi}}$.

Note that the localization of $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ at the cocartesian edges is the actual colimit of the simplicial object it unstraightens, and similarly for $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$. In particular, the category

$$\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E})$$

carries a canonical $S^1$-action. Because we have not quite asked for all the cocartesian edges of $\mathrm{THH}^{epi}(\mathbf{Cat}^{\mathrm{Ex}})$, the extra-structure entailed by the epicyclic refinement is an oplax action of the *Witt monoid* $\mathbb{W}^{\mathrm{op}} := (S^1 \rtimes \mathbb{N}^{\times})^{\mathrm{op}}$ such that the $S^1$-part recovers the previously mentioned $S^1$-action.

We can identify the extra-structure endowed onto a functor $\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})$ which is a fixed-point for the previous (oplax) actions as follows, see Proposition 7.7 and Proposition 7.10 and Proposition 7.15 in the text:

**Theorem 1.10 — Harpaz-Nikolaus-S.** The functors of Proposition 1.8 induce the following equivalences:

$$\left\{\begin{array}{c}\text{Trace-like}\\ F:\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}\to\mathcal{E}\end{array}\right\} \simeq \left\{\begin{array}{c}F:\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})\to\mathcal{E}\text{ which invert}\\ \text{cocartesian edges over surjections in }\Delta^{\mathrm{op}}\end{array}\right\}$$
$$\left\{\begin{array}{c}\text{Cyclic-invariant}\\ F:\mathrm{THH}^\Lambda(\mathbf{Cat}^{\mathrm{Ex}})\to\mathcal{E}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{Fixed points for the }S^1\text{-action on}\\ \text{cyclic-invariant }F:\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})\to\mathcal{E}\end{array}\right\}$$
$$\left\{\begin{array}{c}\text{Cyclic-invariant}\\ F:\mathrm{THH}^{epi}(\mathbf{Cat}^{\mathrm{Ex}})\to\mathcal{E}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{Oplax-fixed points for the }\mathbb{W}^{\mathrm{op}}\text{-action on}\\ \text{cyclic-invariant }F:\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})\to\mathcal{E}\end{array}\right\}$$

via restriction.

In particular, we can also talk about cyclic invariant functors on $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ by simply specifying that they correspond to cyclic-invariant functors on $\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})$ — in fact, there is also a more down-to-earth way of defining this property.

Via the first equivalence, we can transfer the structure of a lax fixed point for the $\mathbb{W}^{\mathrm{op}}$-action to the restricted functor $F:\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}\to\mathcal{E}$. This canonically promotes $F$ to a functor $F:\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}\to\mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})$ valued in *genuine polygonic objects in $\mathcal{E}$ with Frobenius lifts*, which is made concrete in the bullet points below. This is in particular the case of *cyclic K-theory*, the split-fiber of $\mathrm{K}^{\mathrm{lace}}(\mathcal{C},M)\to\mathrm{K}(\mathcal{C})$.

In this thesis, we do not build the category of genuine polygonic objects with Frobenius lifts, nor do we give a precise, functorial proof of Theorem 1.10. This is for a handful of reasons: the first is that the proof (actually, even the statement) involves a lot of extraneous technology about 2-categories that has not been worked out so far, and we have enough on our plate to not have to develop on one's own the theory of 2-categories to the point we need it to be. We note that such attempts at developing this technology have been made in the preprint [AMGR17], to tackle the cyclotomic part of the story, but the proofs therein do not have the degree of precision we strive for.

The second reason is that we want to give a less precise but more low-to-the-ground approach to understand what is going on (and why we need the level of technology that we claim) which will hopefully complement the literature we will later produce and be a more gentle introduction to it. Finally, we note that we will prove those precise statement later, in a series of articles joint with Yonatan Harpaz and Thomas Nikolaus, provisionally cited as [HNS25].

Having said this, if we forget about some of the coherences, here is what this structure entails to: given a cyclic-invariant $F:\mathrm{THH}^{epi}(\mathbf{Cat}^{\mathrm{Ex}})\to\mathcal{E}$, its restriction to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ has the following supplementary features:

- For every $(\mathcal{C},M)$ laced category, $F(\mathcal{C},M^{\otimes n})$ has an action of the cyclic group $\mathrm{C}_n$.
- When $M=\mathrm{id}$, all the possible $\mathrm{C}_n$-actions arise from the restriction of a $S^1$-action
- For every $(\mathcal{C},M)$ laced category and $n,k\geq 1$, there is a canonical $\mathrm{C}_n$-equivariant map
$$F(\mathcal{C},M^{\otimes n})\longrightarrow F(\mathcal{C},M^{\otimes nk})^{\mathrm{hC}_k}$$
- When $M=\mathrm{id}$, the above maps for varying $n$ are given by forgetting the equivariant structure from a $S^1$-equivariant map:
$$F(\mathcal{C},\mathrm{id})\longrightarrow F(\mathcal{C},\mathrm{id})^{\mathrm{h}S^1}$$

The first and third points characterize the datum contained in a genuine polygonic structure with Frobenius lifts, i.e. the lift to a functor $F:\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}\to\mathrm{GenPgc}(\mathcal{E})$, whereas the refined structure on the restriction $\mathcal{C}\mapsto F(\mathcal{C},\mathrm{id})$ is usually known as a genuine *cyclotomic* structure with Frobenius lifts. One can show that every localizing invariant $F:\mathbf{Cat}^{\mathrm{Ex}}\to\mathcal{E}$ gives rises to a cyclic-invariant $F:\mathrm{THH}^{epi}(\mathbf{Cat}^{\mathrm{Ex}})\to\mathcal{E}$ whose restriction to the tangent bundle is given by

$$F^{\mathrm{cyc}}(\mathcal{C},M):=\mathrm{fib}(F(\mathrm{Lace}(\mathcal{C},M))\to F(\mathcal{C}))$$

Thus, the functor on $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ $F^{\mathrm{cyc}}$ inherits extra structure.

Genuine polygonic objects with Frobenius lifts correspond to a special class of genuine polygonic objects which we will again not construct precisely in this thesis; roughly, by post-composing the structural maps by the canonical

$$X^{\mathrm{hC}_k} \longrightarrow X^{\tau\mathrm{C}_k}$$

where $(-)^{\tau\mathrm{C}_k}$ denotes the *proper Tate construction* of $\mathrm{C}_k$, one turns every object of $\mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})$ into a *genuine polygonic object* whose category is denoted $\mathrm{GenPgc}(\mathcal{E})$. This latter category happens to be better behaved relative to linearization, notably because the endofunctor of Sp given by

$$X \longmapsto (X^{\otimes k})^{\tau\mathrm{C}_k}$$

is exact. In particular, one can show that linearization preserves the genuine polygonic structure on $F^{\mathrm{cyc}}$ but in general, kills the Frobenius lifts i.e. the maps to the proper Tate construction of the linearization need no longer factor through homotopy fixed points. This leads to the following theorem, see Theorem 7.18:

**Theorem 1.11 — Harpaz-Nikolaus-S.** Let $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a Verdier-localizing functor. Then, $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}$ upgrades to a functor

$$\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \longrightarrow \mathrm{GenPgc}(\mathcal{E})$$

such that the canonical map $F^{\mathrm{cyc}} \to \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}$ upgrades to a map of genuine polygonic objects, where $F^{\mathrm{cyc}}$ is given its above-mentioned structure with Frobenius lifts.

In particular, the trivial structure functor $\mathrm{triv} : \mathcal{E} \to \mathrm{GenPgc}(\mathcal{E})$ has a right adjoint given by TR. In fact, truncating the genuine polygonic structure to only keep the $n$ first objects, one gets a tower

$$\mathrm{TR} \xrightarrow{\simeq} \lim_{n\in\mathbb{N}^*} \left( \dots \longrightarrow \mathrm{TR}_{[n]}\,\mathrm{trunc}_{[n]} \longrightarrow \dots \longrightarrow \mathrm{TR}_{[1]}\,\mathrm{trunc}_{[1]} \simeq \mathrm{fgt}_1 \right)$$

whose limit is TR. It follows from the Theorem that we have a canonical map

$$F^{\mathrm{lace}}(\mathcal{C}, M) \longrightarrow \mathrm{TR}(\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}(\mathcal{C}, M))$$

natural in $(\mathcal{C}, M)$, which even factors through $F^{\mathrm{cyc}}(\mathcal{C}, M)$. Here $\mathrm{trunc}_{[n]}$ denotes the truncation functor, which we will most often suppress from the notation. We claim that the previous structure on $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}$ allows us to recover the higher derivatives, a surprising phenomenon since there are many functors whose first derivative does not tell you all there is to know about their Taylor tower. This can be seen as a similar phenomenon as the main results of [Goo91], where the key point saving us is that we can interplay the different $\mathcal{C}$ to get this extra-structure.

This is encapsulated by the following result, see also Proposition 7.27 and Theorem 7.28 in the text:

**Theorem 1.12 — Harpaz-Nikolaus-S.** Let $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a finitary Verdier-localizing functor. Then, there is a pullback square

$$\begin{tikzcd}
\mathrm{P}_n^{\mathrm{fbw}} F^{\mathrm{lace}} \arrow[r] \arrow[d] & (\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})^{\mathrm{hC}_n} \arrow[d] \\
\mathrm{P}_{n-1}^{\mathrm{fbw}} F^{\mathrm{lace}} \arrow[r] & (\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})^{\mathrm{tC}_n}
\end{tikzcd}$$

which identifies $\mathrm{P}_n^{\mathrm{fbw}} F^{\mathrm{lace}}$ with $\mathrm{TR}_{[n]}(\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})$. In particular, we find that the previous map

$$F^{\mathrm{lace}} \longrightarrow \mathrm{TR}(\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})$$

is the canonical map of $F^{\mathrm{lace}}$ to the limit of its Taylor tower.

**Corollary 1.13** The limit of the Taylor tower of laced K-theory $\mathrm{K}^{\mathrm{lace}}(\mathcal{C}, M)$ is $\mathrm{TR}(\mathcal{C}, M)$, for $(\mathcal{C}, M)$ a laced category.

In particular, we recover the computation of Lindenstrauss-McCarthy [LM12] of the Taylor tower of K-theory for connective ring spectra. Moreover, this result gives a particularly nice insight on the extra structure on THH:

**Slogan** The (genuine) polygonic structure on $\mathrm{THH}(\mathcal{C}, M)$ encodes precisely the Taylor tower of laced K-theory.

Remark that the $n^{th}$-homogeneous layer of the Taylor tower is given by $\mathrm{THH}(\mathcal{C}, M^{\otimes n})_{\mathrm{hC}_n}$ which precisely categorifies the $n^{th}$-homogeneous piece in the formula given for the (logarithm of the) determinant via traces of iterated powers of Equation 1. In particular, we think about this slogan as some categorified version of a previously stated slogan.

The last question we have not tackled yet is: "When does K-theory coincide with the limit of its Taylor tower?"; in analogy with usual calculus, we will call such a property *analyticity*. The first attempts at understanding the relation between K-theory and TR were undertook by Almkvist, see [Alm78], where he compared in our notation $\pi_0\,\mathrm{K}^{\mathrm{cyc}}(\mathrm{Perf}(R), \mathrm{id})$ and $\pi_0\,\mathrm{TR}(\mathrm{Perf}(R), \mathrm{id})$, which is the Witt vectors of $R$. The former is identified to *rational Witt vectors*. We also point the reader to [DKNP22] and its follow-up [DKNP23] where this comparison is also done with coefficients, and in a language closer to ours.

The upshot of those results is that

$$\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, M) \longrightarrow \mathrm{TR}(\mathcal{C}, M)$$

is not an equivalence when $\mathcal{C} = \mathrm{Perf}(R)$ and $M$ is $R$ viewed as a $R$-bimodule. However, the difference comes down to one side being rational and the other complete with respect to the t-topology. One could hope that if $M$ was more highly connective, then the growth in connectivity of $M^{\otimes n}$ would enforce some sort of nilpotence which makes everything rational.

This is indeed what happens, and we claim this is a very general phenomenon. By working through the ideas of Goodwillie in [Goo91] and adapting them to the higher categorical world, we have the following criterion, see Theorem 9.19 in the text:

**Theorem 1.14** Suppose $\mathcal{C}_{\geq 0} \subset \mathcal{C}$ is a full stable subcategory closed under colimits and $F : \mathcal{C} \to \mathcal{D}$ is a functor to a presentable stable category with a right-complete t-structure compatible with filtered colimits. Suppose that

- $F$ is reduced
- $F$ sends $\mathcal{C}_{\geq 0}$ to connective objects in $\mathcal{D}$
- $F$ preserves sifted colimits of objects of $\mathcal{C}_{\geq 0}$

Then, $F$ converges with the limit of its Taylor tower on $\mathcal{C}_{\geq 1}$, i.e. on suspensions of objects of $\mathcal{C}_{\geq 0}$.

Remark that the first condition is easily enforced: for every $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$, there is a functor $F^{\mathrm{red}}$ such that $F^{\mathrm{red}}(\mathcal{C}, 0) \simeq 0$ and a natural splitting $F^{\mathrm{red}}(\mathcal{C}, M) \oplus F(\mathcal{C}, 0)$; moreover, $F$ and $F^{\mathrm{red}}$ have the same Taylor tower. The second condition is a bit stronger, but can be replaced by sending $\mathcal{C}_{\geq 0}$ to uniformly bounded below objects in $\mathcal{D}$, by shifting the t-structure on the target. The most stringent condition is the third one, and it is usually the hardest to check in practice.

Our goal is to apply this criterion to the functor $\mathrm{K}^{\mathrm{lace}}(\mathcal{C}, -)$. For this, we have to find a special subcategory of *non-negative* bimodules and show that $\mathrm{K}^{\mathrm{lace}}$ satisfies the third condition.

If $\mathcal{C}$ is the stable envelope of an additive category $\mathcal{A}$, then $\mathcal{C}$-bimodule correspond to direct-sum preserving functors $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Sp}$ and there is a very natural full subcategory closed under colimits of this: those functors that land in connective spectra. In fact, in this case, this category is the connective part of a t-structure on the category of $\mathcal{C}$-bimodules.

More generally, we have shown the following in [Sau23b, SW25], which is also cursorily explained in Corollary 10.11, Theorem 10.15 for the first part and Lemma 10.21 for the claim about Lace:

**Theorem 1.15 — S.-Winges.** Let $\mathcal{E}$ be a weakly-idempotent exact category. Then, specifying an equivalence $\mathcal{C} \simeq \mathrm{Stab}(\mathcal{E})$ between a stable category and the stable envelope of $\mathcal{E}$ is the same thing as specifying a *bounded heart structure* on $\mathcal{C}$ whose heart is $\mathcal{E}$, i.e. there is an equivalence of categories

$$\mathbf{Exact}_\infty^{w-i} \simeq (\mathbf{Cat}^{\mathrm{Ex}})^{bdd-\heartsuit}$$

between the category of weakly-idempotent exact categories and the category of stable categories equipped with a bounded-heart structure and functor preserving the heart.

Moreover, if $\mathcal{C}$ has a bounded heart structure and $M$ is a $\mathcal{C}$-bimodule that sends $\mathcal{E}$ to connective spectra, then $\mathrm{Lace}(\mathcal{C}, M)$ inherits a bounded heart structure.

Here, the weakly-idempotent condition on $\mathcal{E}$ is asking that every retraction of $\mathcal{E}$ splits: if the retraction has a fiber or a cofiber, then this is imply by the splitting lemma but there is not in general enough fibers in a general exact category $\mathcal{E}$ to make this work.

Note that if $\mathcal{E}$ is only exact, it need no longer be true that those $\mathrm{Stab}(\mathcal{E})$-bimodule which send $\mathcal{E}$ to connective spectra form the connective part of a t-structure. Indeed, the operation $\iota_{\geq 0}\tau_{\geq 0} : \mathrm{Sp} \to \mathrm{Sp}$ which truncates and then re-includes into the category of spectra only preserves direct sums, but not exact sequences in general. Still, such bimodules do form a well-defined category $\mathrm{Bimod}(\mathrm{Stab}(\mathcal{E}))_{\geq 0}$ which is closed under colimits.

We are able to show that both $\mathrm{K}^{\mathrm{lace}}$ and $\mathrm{THH}^{\mathrm{lace}} := \mathrm{THH} \circ \mathrm{Lace}$ have a (concrete) extension to exact categories which send the map $\mathcal{E} \to \mathrm{Stab}(\mathcal{E})$ to an equivalence. This affords models that are easier to manipulate and, via the only argument so-far which actually uses a concrete construction of the above functors, we can show the following, which is spelled out in Proposition 11.2:

**Theorem 1.16** Suppose $\mathcal{C}$ has a bounded heart structure and denote $\mathrm{Bimod}(\mathcal{C})_{\geq 0}$ the category of those bimodules $M$ which send the heart to connective spectra.

Then, both $\mathrm{K}^{\mathrm{lace}}(\mathcal{C}, -)$ and $\mathrm{THH}^{\mathrm{lace}}(\mathcal{C}, -)$ verify the hypotheses of Theorem 1.14 up to the remarks formulated immediately afterwards.

**Corollary 1.17** If $\mathcal{C}$ has a bounded heart structure such that $M$ sends the heart to 1-connective spectra, the map

$$\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, M) \longrightarrow \mathrm{TR}(\mathcal{C}, M)$$

is an equivalence.

To get to the final result of this thesis, it remains to explain what is the Taylor tower of $\mathrm{THH}^{\mathrm{lace}}(\mathcal{C}, -)$. By previous considerations, $\mathrm{THH} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ lifts to a category of genuine cyclotomic spectra. Any genuine cyclotomic spectra can be viewed as a genuine polygonic object and this functor admits a right adjoint denoted $R$. One can show (but we won't do it precisely in the thesis) that the Taylor tower of $\mathrm{THH}^{\mathrm{lace}}$ is given by $R\,\mathrm{THH}(\mathcal{C}, M)$, where $\mathrm{THH}(\mathcal{C}, M)$ is endowed with its usual[3] genuine polygonic structure.

In particular, applying TC to the Taylor tower of $\mathrm{THH}^{\mathrm{lace}}$ recovers TR, which is also the Taylor tower of K-theory. Hence, when $\mathrm{K}^{\mathrm{lace}}$ and $\mathrm{THH}^{\mathrm{lace}}$ are analytic, we have the following, which is Theorem 11.4 in the text:

**Theorem 1.18** Let $\mathcal{C}$ be a bounded heart category and $M$ a $\mathcal{C}$-bimodule which carries the heart to $\mathrm{Sp}_{\geq 1}$. Then,

$$\begin{tikzcd} \mathrm{K}(\mathrm{Lace}(\mathcal{C}, M)) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{Lace}(\mathcal{C}, M)) \arrow[d] \\ \mathrm{K}(\mathcal{C}) \arrow[r] & \mathrm{TC}(\mathcal{C}) \end{tikzcd}$$

is a pullback square.

[3]Here, "usual" is in the sense of we defined it a couple of Theorems ago

This is a generalization of the split-square zero case of the main result of [DGM13], which we have recalled previously in the introduction, and taking the above structure to be a weight structure, we also get a generalization of the split-square zero case of [ES21]. In particular, we stress that for rings and connective bimodules, the above gives the split square-zero case of Dundas-Goodwillie-McCarthy thanks to the shift in the identification between $\mathrm{Perf}(R \oplus M)$ and $\mathrm{Lace}(\mathrm{Perf}(R), \Sigma M \otimes_R -)$.

## 1.4 Glimpses of the future

There are still many open questions and directions to pursue the above works. Let us go through them, loosely ordered by either urgency or simplicity.

Theorem 1.18 is about *split* square-zero extensions, but the full Dundas–Goodwillie–McCarthy usually involves about nilpotent extensions, i.e. maps $A \to B$ of connective ring spectra which are surjective on $\pi_0$ with nilpotent kernel, the case of non-necessarily split square-zero extension being the case where the ideal $I^2 = 0$. It is less clear what is supposed to play the role of general nilpotent extensions of bounded heart categories: there is an abstract definition, following the general theory of square-zero extensions as in [Lur17a, Section 7.4].

In a more concrete approach, [ES21] also have a concrete definition for such extensions, at least when restricted to bounded weight structure. We do not know whether the abstract and the concrete definition coincide; moreover, the concrete approach does not quite track to a similar proof of Dundas-Goodwillie-McCarthy, though we have been made aware of on-going work of Levy and Sosnilo tackling this direction.

Another direction follows the ideas introduced by Efimov in [Efi24], and pertains to the extension of non-connective K-theory and more generally, localizing invariants to dualizable categories. The machinery of the present text is geared towards connective K-theory because $\mathrm{K}^{\mathrm{lace}}$ has a universal property in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, which is not the case of $\mathbb{K} \circ \mathrm{Lace}$; in fact, the latter need not even be localizing because Lace does not preserve Karoubi sequences (in fact even some Verdier sequences, as is explained in the text).

In on-going work, we want to investigate the trace-methods questions of dualizable category: what is the tangent bundle of $\mathrm{Pr}^{\mathrm{dual}}$, what is $\mathrm{Lace}^{dual}$, its square-zero extension functor, and how much of the theory changes in this set-up. Early parts of the investigation suggests although there is an equivalence

$$\mathrm{T}_{\mathcal{C}}\mathrm{Pr}^{\mathrm{dual}} \simeq \mathrm{End}^{\mathrm{L}}(\mathcal{C})$$

for $\mathcal{C}$ dualizable, so that in particular Ind preserves the tangent category, the dualizable square-zero extension funcotr $\mathrm{Lace}^{dual}(\mathcal{C}, M)$ is almost never compactly-generated even though its compact objects are $\mathrm{Lace}(\mathcal{C}^{\omega}, M)$ so the theory does differ. We do not quite know whether $\mathrm{Lace}^{dual}$ sends Karoubi sequences of laced-dualizable categories to Karoubi sequences in $\mathrm{Pr}^{\mathrm{dual}}$: by as-of-yet unpublished work of Efimov, this is akin to ask whether $\mathrm{Lace}^{dual}$ will behave closer to $\underline{\mathrm{Map}}_{\mathrm{Pr}^{\mathrm{dual}}}(\mathrm{Perf}(\mathbb{S}[t]), -)$ or $\underline{\mathrm{Map}}_{\mathrm{Pr}^{\mathrm{dual}}}(\mathrm{Nuc}(\mathbb{S}[t]), -)$. In his work, Efimov shows the former need not preserve Karoubi-sequences but the second does; moreover, the Karoubi-approximation of the first when passing to localizing motives is given by the second and coincides with curves on K-theory (plus a constant term), as defined for instance in [McC23] which also showcase that there are trace methods ideas that apply to this functor.

Whether $\mathrm{Lace}^{dual}$ has the wanted property or not, it is reasonable to expect trace-methods to give new and interesting results to compute $\mathbb{K}(\mathcal{C})$ for $\mathcal{C}$ a genuinely new object (dualizable non-compactly generated) that is not captured by usual small trace-methods.

We also want to note that in the not-yet-publicly-available [HNS23], two of the coauthors of some of the material presented in this text as well as Jay Shah, have undertook the study of hermitian trace methods, following the bulk of work done in [CDH$^+$23a, CDH$^+$23b, CDH$^+$23c] to push the hermitian story to a similar realm as the algebraic K-theory one. This study has evolved to resemble a lot in the techniques the ones used in here, though the author admits to lack the precise knowledge of what changes in the Poincaré setting.

Finally, there are obvious constructions and candidates for theories of non-commutative motives in trace-methods. The landscape in non-commutative motives has changed dramatically as this manuscript was being written: the main theorem of [RSW25] shows that the category of localizing motives is actually a localization of $\mathbf{Cat}^{\mathrm{perf}}$ and in upcoming work, Efimov has managed to compute mapping spaces of localizing $\kappa$-motives (this is related to a previous point) as well as showing that they form a large rigid category.

The perspective of the former is quite interesting: in the trace-methods world, quite many conditions were phrased purely in terms of inverting a collection of arrows (see all the variants of cyclic and trace invariance), leading to some trace-like motives, and one can further add adjectives to give different flavours to the theory. In particular, one of the main results of Ramzi's thesis [Ram24a] shows that the category of trace-invariant, fiberwise-colimit preserving functors is precisely given by Sp (or $\mathrm{Sp}^{BS^1}$, GenCyc(Sp) when trace-invariance is understood in some $\mathrm{THH}^{symbol}(\mathbf{Cat}^{\mathrm{Ex}})$ and not just in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$). However, when removing or altering some adjectives in the above, the result is no longer known and it is yet unclear

## 1.5 Acknowledgments.

The acknowledgments of this thesis are given by the initial set of acknowledgments receiving a name-preserving morphism from the list of people who have helped the author through his mathematical and non-mathematical journey (suitably completed at every forgotten memory). Note that in this manuscript, all acknowledgments are supposed to be well-written which is not restrictive since there is a functorial, colimit-preserving way of achieving this, often called *proof-reading* — we will always implicitly apply this functor to all our acknowledgments.

Even though his commitment to the joke tempts the author to not say more, below we sketch a construction that the set of acknowledgments characterized by the previous property does indeed exist. This is done by brute force (and faltering memory).

As any good list of names in mathematics, the following ordering is alphabetical. Souvent, on regroupe d'abord les gens sous un groupe commun (e.g. 'PhD students') and then order their names; also the language is adapted et change de manière fluide to best suit la personne visée. Tout oubli is fully la responsability de l'auteur et il s'en excuse wholeheartedly. Leider ist meine Deutsch nicht gut genug, um mehr als diese Worte zu schreiben.

In **Bonn**, I want to thank Christian Kremer, Dominik Kirsten, Kaif Hilman and Jonas McCandless for their warm welcome, their exchange of ideas and the many moments that made me feel for the first time that I was part of a community of mathematicians across the world.

Parmi les **Doctorants de l'équipe TA**, il me faut citer Benachir El Allaoui mais qu'à partir de 14h, Coline Emprin, Oisin Flynn-Connolly with whom I spent so long chatting about every possible subject between the coffee machine and my desk so much so that I would look up the news in the morning just to prepare those moments (why did they reelect Trump when you left, we could have said so much ...) — also, I must thank him for the name *laced categories*; what a find!, Nic□las Guès pour les cubes et les cubes, Marcus Nicolas pour pour sa capacité sans fin à poser des questions intéressantes, Hugo Pourcelot pour l'accueil et les conseils prodigués en début de thèse, Francesca Pratali for her amazing positive presence and for the maths chat (the questions you asked but also, near the end, the questions that I asked!), Dominik Schrimpel for having lighted up the office during this fourth year, including (but not limited to) maths discussion, Jakob Ulmer for a similar deed during the second and third year, as well as the few moments when he spoke to me in German without realizing (I'm afraid this is going to be much more common soon), mais aussi Victor Luca i Rocio et Guillaume Laplante-Anfossi.

Et parce qu'il n'y a pas que les maths dans la vie (il y a aussi l'innomable jeu de cartes), l'équipe que je surnomme **Doctorants Fun**, où il me faut remercier l'inénarrable Wassim Aboussi, Arthur Arnoult pour son courage sans faille (et sa maison aussi, enfin j'espère pour elle) ainsi que les confitures de sa maman, Maissâ Boughrara pour ses nombreuses bonnes adresses de restaurant dans Paris qui sont parfois un tout petit peu trop populaires, Elie Cerf malgré le fait que j'ai dû me coltiner Lucas tout seul pendant la dernière année parce que môssieur ne voulait pas faire une sixième année, Nicolas Conanec à qui on a déjà assez fait de blagues sur la Bretagne comme ça mais attention, il pleut sur ta gauche, Loth Damagui Chabi pour ta ferveur supportrice (le PSG a gagné la ligue des champions avant que je parte, c'est fou), Joelle Ishak oenologue de renom et pétrie de qualités que je ne listerais pas pour ne pas faire de remarques superfétatoires, Lucas Lagarde pour une quantité de blague à la limite de l'intankable, des gâteaux d'une saveur sans

pareille (pardon Grégory) et ... ah j'ai un appel, oui allô, c'est les remerciments, ils ont eu un grave accident d'écriture, ils ont besoin de 50€ pour être terminés (si tu relis ce message dans 10 ans, change dans ta tête pour le shitpost du moment) — Julien Malartre qui est la seule autre personne à avoir jamais suggéré d'aller manger avant 12h (merci!), David Nahmani à qui je lègue le flambeau de Yu-Gi-Oh dans le groupe des doctorants (standby phase shifteur dimensionel, réponse?), Neige Paulet , Romain Pinguet qui malgré tous ses efforts n'a pas réussi à sortir de l'équipe doctorants fun, René Pfistcher for such a positive attitude towards math and life in general, Mathieu Vallée qui est le seul doctorant non-mathématicien de cette liste, ce qui confirme un certain tropisme, et enfin Vincent Viau à qui je souhaite de pouvoir continuer à pisser dans la Garonne, à collectionner des cartes Pokémon et dont les pendus vont me manquer, surtout quand ils sont interrompus de manière inopiné. Enfin, j'ajoute à la liste Alice Contat et Vadim Lebovici qui sont à la limite de passer dans la team permanents (bravo encore) mais qui sont encore trop fun pour le moment.

In a team with significantly less members, but with an impact rivaling the others, from the **ERC group**, I want to thank wholeheartedly Ran Azouri (but with the same caveat as Benachir, see *doctorant TA*), Jordan Levin for three incredible years where I saw you become a fully-fledged mathematician and I got to enjoy your remarkable company in the office and in our many trips across Europe, how boring would those three years have been without your presence, Tasos Moulinos whose time among us was cut short (but for happy reasons), and Guglielmo Nocera for his upbeat attitude and his undaunted courage which led us to a wonderful paper on the assembly map of bordism-invariant functors, which I can guarantee would not exist without your enthusiasm and your fervor.

Dans l'équipe **Equipe Topologie algébrique** (oui c'est redondant mais dans cette équipe, on est attaché à se rappeler quand on a des trucs en double), je veux remercier Gabriel Angelli-Knoll for many coffee-machine chat and who also gets the privilege to top yet another list of names, Christian Ausoni pour son attitude bienveillante qui m'a fait rencontrer Bjorn et m'a toujours soutenu, Charles De Clerq dont je speedrun les remerciements sans pourtant autant oublier nos discussions autour d'un café, Geoffroy Horel pour les échanges mathématiques autour d'un café (il y a quelque chose avec le café dans ce labo, vous remarquerez), Anne Quéguiner-Mathieu et enfin Bruno Vallette qui a toujours eu une anecdote ou un conseil à partager, et qui détient le record d'être le seul à s'être jamais plaint de mon anglais en exposé. Les attentifs qui auront remarqué qu'il manque deux joyeux larrons sont invités à lire le paragraphe suivant.

Dans ce qui pourrait s'apparanter à une sous-équipe de la précédente, mais qui mérite d'être dans une *league of its own* comme ils disent de l'autre côté de la Manche, l'équipe **Equipe TA, Café, Chocolat et autres douceurs** composé de deux membres, je veux remercier tout particulièrement Grégory Ginot, à qui il m'est impossible de faire une liste complète tant elle serait longue de toutes les fois où j'ai été ravi de le voir parce qu'il avait une tarte, un gâteau, du chocolat ou une bonne nouvelle administrative à m'annoncer, à qui je dois mon introduction à la topologie algébrique (et donc toute la suite) à une époque où le mot pandémie pouvait encore être confondu par certains avec une maladie des pandas, et Eric Hoffbeck, l'homme qui murmure (avec Maher) à l'oreille des machines à café, qui a aussi un stock sans fin de chocolat et qui a été à l'origine des conseils les plus précieux que j'ai pu recevoir durant ma thèse.

Revenons à une équipe dont l'ancienneté fera palir toutes les autres, baptisé sobrement **LLG**, dans laquelle je mets Nicolas Guérin, que les dieux ont généreusement décider de placer dans le même RER B un jour de septembre 2015, avec qui on a fait les Cassini, les colles d'anglais, la boue de la Courtine et la grèle sous la tour Eiffel, sans qui je serais un bien moindre mathématicien (du genre qui simplifie dans $\mathcal{L}(E)$) mais probablement aussi un être humain beaucoup moins intéressant, Jean Rubin à qui aucun problème ne résiste (et pourtant, j'ai du écumer tous les magasins de casse-tête de la capitale pour essayer) et qui a probablement fait reculé la science de 50 ans quand il a décidé de ne pas faire une carrière en combinatoire (mais dont la contribution à l'INSEE a certainement compensé pour ce qui en est de la société), Olivier Salagnac qui a bravé la 5/2 pour rejoindre le groupe (ou plutôt pour garder son groupe de volleyeurs de la cours Molière préféré) et sans lequel je ne pense pas que j'aurais pu supporter Nicolas dans sa phase shark. Je n'oublie pas non plus Chloé, Foulques, Félix, Maxime, Téodora et Orso comme membre honoraires, bien qu'ils soient nommé dans d'autres équipes.

Dans l'équipe **Maman** qui ne comporte qu'elle, mais qui gagnerait avec aise contre toutes les autres, je me dois de remercier une personne dont l'amour, la générosité et le soutien constant sont autant de piliers sur lesquels je me suis appuyé.

And now for a team whose name is a mouthful, **Mathematicians from afar but not Bonn or Münster**, whose contributions and/or discussions have influence either directly or indirectly this manuscript, I thank Dhyan Aranya in Amsterdam, Baptiste Calmès in Lens though I've never met him there, Emanuele Dotto in Warwick though Venice is truly the reason he is on this list (and the math it allowed to happen there, this is not just to be reinvited), Fabian Hebestreit in Bielefeld though this is written certainly before a lot of the anecdotes I would want to share have happened, Ishan Levy when we met at YTM Münster, Zhouhang Mao in many places for the countless questions and the sharp wit that has forced me to clarify and clear mistakes from my mind so many times, and finally, there was almost a Regensburg team but they had the good idea of sharing a general position in the alphabet, Pier Federico Pacchiarotti with an asterisk for Warwick, Vova Sosnilo for countless enlightening discussion and Christoph Winges for an extremely fruitful collaboration and just as much, if not more, ideas and words of mathematical wisdom.

The final team of maths people is **Mathematics Münster** where I want to thank Arthur Bartels for his time chatting about assembly maps, Thomas Nikolaus which has been so important to the realization of this very thesis and whose insight is almost unparalleled, Phil Pützstück who has spotted more typos in this draft than anyone else and has generally been an incredibly nice person to chat maths with, et finalement Maxime Ramzi, pour toutes les discussion sur les méthodes de traces qui ont aidé ce manuscript à grandir mais aussi pour toutes les autres échanges et les moments partagés, la liste desquels je suis sûr que les années qui viennent ne feront qu'augmenter.

Another team without which this manuscript would be but a shadow of what it is are the two **Referees**, namely Bjorn Dundas who is one of the kindest persons I have ever met, has shared so many insights throughout our exchanges and to whom I still owe a drink of his choosing for forcing to write recommendation letters in a hurry, as well as Marco Robalo who bombarded this document with questions and remarks, improving tremendously the presentation and the clarity of it — if you, reader, have an easier time going through it, it is thanks to him first and foremost.

Bien que leur contribution risque d'être effacé par l'histoire avec un grand H (pas celle qu'on écrit maintenant, donc), l'équipe **Restaurant**, composée de l'indéboulonable Sandwich Poulet (*pouler* ?) Mayo et de la boulangerie de Villetaneuse, qui a nourri l'auteur pendant plus d'un an lorsque que le restaurant adminstratif a failli à ses devoirs (et c'est long, une année de poulet mayo ...)

A une équipe dont l'importance n'est pas compréhensible par qui n'a pas dû affronter les pièges de l'administration française, les personels du **Secrétariat et Administration**, Leila pour sa patience infinie et sa capacité à m'aider alors même qu'elle devait apprendre les démarches quasiment en même temps que moi, Monia pour m'avoir sortir du pétrin pour plusieurs missions, Yolande pour sa gentillesse sans fin et les nombreuses discussions et les blagues qui ont animés chaque jour de ma présence au LAGA, ainsi que Jean-Philippe et Sembala, le premier pour sa connaissance si fine de l'adminstration française dont un diplôme ès *chasse au pigeon* et sa capacité à la partager et le second pour sa patience et son soutien constant quand les choses ne marchent pas parce que les dieux de Notilus étaient fâchés.

Enfin, une équipe au nom si court mais que j'apprécie si particulièrement, les **X** et notamment ceux de la sections volley: Léo Baujeu qui m'a accueilli tant de fois avec Paul pour d'innombrables jeux de sociétés et autres voyages en Ecosse, Guillaume Bourgarel que je suis toujours ravi de recroiser dans Paris lorsqu'il le propose, Elias Khallouf que j'ai toujours plaisir à recroiser aléatoirement dans Paris au moment où je m'y attends le moins, Maixent Esmieu-Fournel pour toutes les blagues et les moments de rigolade, Tom Jacquin avec qui j'ai partagé tant de parties de volley endiablés parfois jusqu'au bout de la nuit (du volley (et littéralement, en plus)) et dont l'influence sur mon humour a été significative (heureusement ou malheureusement, je laisserais l'intéressé en décider), Rind El Hage pour laquelle I do not know whether je devrais écrire en anglais or in

French, but in any case, a toujours le smile (celui-là marche dans les *deux* langues!) et à qui je souhaite autant d'aventures que Zaki (c'est un écocup dans ta main?), Yahya Ibenbrahim qui est tout aussi responsable que Tom pour certains traits d'humour ainsi que pour beaucoup de courbatures suites aux fameux volleys endiablés, Foulques Renard dont la curiosité n'a d'égale que sa capacité de partage et qui continue de m'impressioner par sa faculté à se renouveler à chaque fois que je le revois, Matthieu Rykner dulequel je suis très heureux d'avoir su me protéger de l'influence dans le cadre de l'humour mais avec lequel je suis aussi très content de partager les boirs et le déboirs rugballistique du stade, du XV de France et plus généralement du sport français (et bientôt allemand, qui sait) — attention Rind a un écocup dans la main par contre, Michel Thomazo pour lequel j'ai très peur de me planter sur le nombre d'enfants quand ces pages sont écrites mais qui a toujours été si gentil et drôle que ce soit au volley ou ailleurs, Olivier Truffinet dont les anecdotes légendaires ont égayés moultes conversations (même s'il faut lui reconnaître un certain assagissement salutaire ces derniers temps) et qui ne cesse d'impressioner par sa curiosité et son intérêt pour des mathématiques que je croyais impénétrables pour les non-initiés, Chloé Vernière, qui a (!!!) et qui ne cessera certainement jamais d'avoir que ce soit pendant les volleys improvisés sur la plage ou toute autre activité sportive ou non sportive. Je n'oublie pas Félix aussi, qui est de l'année d'après ainsi que Nicolas, Jean et Geoffroy qui sont membres honorifiques de l'équipe, et ont déjà été cités (désolé Olivier, peut-être en 7/2, s'il fait pas trop chaud pour l'oral de Chimie, mais l'ENS c'est pas si mal non plus).

Finally, in the only other team with a single member, **Yonatan Harpaz** (whose first name begins by a very late letter in the alphabet so that I can put him in the rightful place without breaking this *listing stuff in alphabetical order to pretend everyone contributed the same* which is so dear to mathematicians), whose only member is Yonatan Harpaz (unsurprisingly). This is certainly the hardest one to write: I did not have many dreams throughout my childhood — I was more of the small goals philosophy, slowly grinding through the steps — but I distinctly remember more than ten years ago (ouch) thinking "I'd like to be a mathematician", which was quite an incongruous dream back at the time, maybe even more because I understood it as "contributing to research" aka "making a contribution of my own to this great thing which is the combined knowledge accumulated by humankind over many centuries". I don't think I quite believed in it back then.

And through this document, but also the five preprints/publications and the many more that are in preparation, this dream is about to become a reality and the prime contributor to this is none other than you, Yonatan. During those four years, we exchanged so many emails, so many ideas, we chatted about so many topics and you helped me every single time refine my understanding, grow as a mathematician, as a scholar (the word is big but I can't quite find something of the same flavour right now). You were kind, caring and always pushed me in the right direction. I still vividly remember when the project that ended up being this thesis but also our joint work with Thomas and now Maxime started, in January of the first year, while we were in the restaurant administratif of Paris 13. The spark that happened then was almost magical, and I am incredibly proud of where we have carried it and incredibly thankful of how much love and care you poured into the project to make it what it is. I would never have been able to reproduce this, even with a thousand year to spare.

For this, but also for many others things, some that I forgot and some that I do not have the place to list, I extend to you my gratitude, and I hope that in the many years to come, which I hope will be filled to the brim with mathematics, that we will get to experience again those moments, from sharing ideas on the board to the spark during lunch. Thank you.

# 2 Liminary remarks

The goal of this section is *not* to make a complete list of the statements that will be needed throughout this thesis. Instead, we simply take some time to explain a couple of prerequisites results, which hopefully will ease the reader into the following sections. In some sense, this would be the introduction, if introductions were not as standardized and forced to be full of spoilers as they are nowadays (the adjective *liminary* means placed at the beginning). In particular, we omit every proof.

## 2.1 A few words about categories and those that are stable

Let $R$ be a discrete ring. There are a number of categorical gadgets one can associate to $R$; to cite a few classical ones: $\mathrm{Proj}(R)$, the 1-category of finite-type projective modules, $\mathrm{Mod}(R)^{\heartsuit,ft}$, the 1-category of finite-type modules and $\mathrm{Mod}(R)^{\heartsuit}$, the 1-category of $R$-modules[4]. These have increasingly good categorical properties, at the price to have more complicated objects inside of them.

The 1-category $\mathrm{Proj}(R)$ is *additive*: the product and coproduct coincide and the structure of commutative monoid this induces on $\mathrm{Hom}(P,Q)$ via

$$+\colon \mathrm{Hom}(P,Q)\times\mathrm{Hom}(P,Q)\simeq\mathrm{Hom}(P,Q\oplus Q)\xrightarrow{\nabla_*}\mathrm{Hom}(P,Q)$$

is actually that of an abelian group. The 1-category $\mathrm{Mod}(R)^{\heartsuit,ft}$ is also additive and the embedding $\mathrm{Proj}(R)\to\mathrm{Mod}(R)^{\heartsuit,ft}$ preserves direct sums, but there is more: $\mathrm{Mod}(R)^{\heartsuit,ft}$ has all finite limits and colimits and satisfies the axioms of an *abelian category.* Note that if

$$0\longrightarrow P\longrightarrow Q\longrightarrow S\longrightarrow 0$$

is a short-exact sequence of $\mathrm{Mod}(R)^{\heartsuit,tf}$ where $P,S\in\mathrm{Proj}(R)$ then $Q\in\mathrm{Proj}(R)$ and the sequence splits, so that $Q\simeq P\oplus S$ non-canonically. In particular, both $\mathrm{Proj}(R)$ and $\mathrm{Mod}^{\heartsuit,tf}(R)$ have the structure of an exact category, as introduced by Quillen in [Qui73]. Finally, $\mathrm{Mod}(R)^{\heartsuit}$ is also abelian, but also has filtered colimits — in fact, it is generated by $\mathrm{Mod}^{\heartsuit,tf}(R)$ under them with minimal hypotheses on $R$ and thus is a locally-presentable 1-category.

We will discuss more lengthily exact categories later in the text. Briefly, Gabriel-Quillen's embedding theorem affords us the following perspective on exact categories: an exact category $\mathcal{E}$ is an additive full subcategory of an abelian category $\mathcal{A}$ closed under extensions. Those sequences in $\mathcal{E}$ that are short-exact when viewed in $\mathcal{A}$ correspond to a choice of a special class of exact sequences and they necessarily include all the split ones.

There are two extremal possibilities for exact categories: one is to choose all of the abelian category as a full subcategory and the other to pick something like $\mathrm{Proj}(R)$ inside $\mathrm{Mod}^{tf}(R)$, i.e. a full subcategory where all the chosen exact sequences split. It is always possible to realize an additive category $\mathcal{A}$ as embedded as a full subcategory of an abelian category where all the special-exact sequences split.

This was the extent of categorical gadgets associated to a discrete ring $R$. But there is more: any ring $R$ can be viewed as a discrete $\mathbb{E}_1$-ring spectrum. This is trickier than it seems because two different phenomenon are happening at once. The central point of $(\infty,1)$-categories is that there is a well-wrought theory where (homotopy types of) spaces are the freely generated objects of the theory, in the place of sets. In this world, commutative monoids in sets become $\mathbb{E}_\infty$*-monoid in spaces* and abelian groups those $\mathbb{E}_\infty$-monoids in spaces which are *grouplike*. A result of May characterizes those as infinite loop spaces, and those can be further seen as *connective spectra*.

The trick is of course in the adjective: there are some spectra which are not connective. Indeed, in the category $\mathrm{Sp}_{\geq 0}$ of connective spectra, the suspension functor $\Sigma$ is fully-faithful and presentably making it invertible produces a larger category, Sp, of all spectra — doing it presentably has even created some interesting although strange objects, like the spectrum KU whose homotopy groups are periodic. For categorical reasons, Sp is nicer than its full subcategory $\mathrm{Sp}_{\geq 0}$. Hence, when choosing what to replace the category of abelian groups Ab, the standard example of an abelian category, we are given two choices and one is "better" than the other by that Grothendieck quote — it has worse objects but better categorical properties.

On the side of categories associated to a ring object $R$ in Sp: the situation is as follows. If $R$ is a connective ring spectrum, then there is a category $\mathrm{Proj}(R)$ which is additive in the higher sense i.e. where mapping objects are required to upgrade to connective spectra instead of abelian groups and two categories $\mathrm{Perf}(R)_{\geq 0}$ and $\mathrm{Mod}(R)_{\geq 0}$ — where Perf has taken the place of $\mathrm{Mod}^{\heartsuit,ft}$ for ontological and historical reasons — which are *prestable*. In particular, they have all the finite limits and colimits and they quite nicely behaved. The prototypical example of a prestable category

[4] Make a choice of your favorite direction for modules (either left or right) and stick to it.

is $\mathrm{Sp}_{\geq 0}$ which is precisely the free prestable category on one object, namely the connective ring spectrum $\mathbb{S}$, i.e. the sphere spectrum.

Of course, our terminology suggests that if $R$ is any ring spectrum, there should also be two categories $\mathrm{Perf}(R)$ and $\mathrm{Mod}(R)$ which ought to be simply *stable*. And there is, as is discussed in Section 1 of [Lur17a]! Note that Barwick has also developed the higher analogue of exact categories in [Bar16] and Klemenc has shown the Gabriel-Quillen embedding theorem holds in this context, see [Kle23], so that we can have the same discussion about exact categories. Note however that we have lost something in the way: there need not be an additive equivalent of $\mathrm{Proj}(R)$ in this more general setup, unless $R$ is connective. Of course, one can still talk about the smallest additive subcategory of $\mathrm{Perf}(R)$ containing $R$ and closed under retracts but this no longer generates $\mathrm{Perf}(R)$ as an additive subcategory — in fact it may be that some exact sequences in $\mathrm{Perf}(R)$ between objects of this subcategory do not split so the name *projective* would be undeserved. In fact, by virtue of a theorem of Sosnilo, $\mathrm{Proj}(R)$ generates $\mathrm{Perf}(R)$ as a stable subcategory if and only if $R$ is a connective ring spectra.

Let us summarize what we have discussed. Let $\mathbf{Cat}^{\mathrm{Ex}}$ denote the category of stable categories and exact functor, $\mathbf{Exact}_\infty$ the category of exact categories and exact functors between them and $\mathbf{Cat}^{\mathrm{add}}$ the category of additive categories and additive functors between them. We have two forgetful functors, the former being fully-faithful

$$\mathbf{Cat}^{\mathrm{Ex}} \longrightarrow \mathbf{Exact}_\infty \longrightarrow \mathbf{Cat}^{\mathrm{add}}$$

Both functors have left adjoints, denoted respectively Stab and $(-)^{\oplus}$. Moreover, to each ring spectrum $R$, we can associate a category $\mathrm{Mod}(R)$ of $R$-modules. This category is presentable stable and compactly generated by its compact objects $\mathrm{Perf}(R)$. In the case where $R$ is connective, then additionally there is an additive full subcategory $\mathrm{Proj}(R)$ of $\mathrm{Perf}(R)$ such that $\mathrm{Stab}(\mathrm{Proj}(R)^{\oplus}) \simeq \mathrm{Perf}(R)$.

There is an operation on rings we have not yet discussed and that will be of critical importance for us. If $R$ is a ring and $S$ a subset of $R$, then one can form $p : R \to R[S^{-1}]$, the localization of $R$ at $S$. This is characterized by a universal property, namely

$$\mathrm{Hom}(R[S^{-1}], Q) \xrightarrow{p^*} \{f \in \mathrm{Hom}(R, Q) \mid \forall, t \in f(S), t \text{ is invertible}\}$$

There is a similar concept for ring spectra, and naturally for stable categories. Let us already remark that the interplay between those notions is way more complicated than the words *similar concept* could lead one to believe.

Let us exclusively focus on stable categories for a moment. The notion of a localization of stable categories $p : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ at a collection of arrows $\mathcal{W}$ is well-defined up to set-theoretical issues, which we eagerly suppress. The following holds — see the appendix of [CDH⁺23b] for a more complete discussion (and proofs!):

**Proposition 2.1** Let $i : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. Then, its cokernel in $\mathbf{Cat}^{\mathrm{Ex}}$ is given by the localization $p : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ at the collection of arrows $X \to Y$ whose kernel is in the image of $i$.

Arrows exhibiting their target as a cokernel in $\mathbf{Cat}^{\mathrm{Ex}}$ are called *Verdier projections*. Taking the kernel of a Verdier projection $p : \mathcal{D} \to \mathcal{E}$ yields a full subcategory $i : \mathcal{C} \to \mathcal{D}$ which is further closed under retracts in $\mathcal{D}$; such arrows are *Verdier inclusions*. The two classes determine one another, that is to say that the projection associated to a Verdier inclusion $i$ obtained as the kernel of $p$ is exactly $p$. In another, fancier way: these are normal mono/epi-morphisms of $\mathbf{Cat}^{\mathrm{Ex}}$, though we note that not all of the categorical epimorphisms are of this form. In particular, they give rise to the notion of a Verdier sequence, which is exactly a null-composite sequence (this is a condition in $\mathbf{Cat}^{\mathrm{Ex}}$):

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

where $i, p$ are associated Verdier inclusions/projections. In particular, Verdier sequences are exactly the fiber-cofiber sequences of $\mathbf{Cat}^{\mathrm{Ex}}$. This structure is quite peculiar, as it seems to combine both some abelian and stable features and certainly we expect in the future that the correct notion of a 2-exact (or 2-stable or 2-abelian) $(\infty, 2)$-category, which is as of yet to be invented, has this as a prime example.

There are two related classes of sequences that will also interest us: on one hand, the *Karoubi sequences* are the related notion in $\mathbf{Cat}^{\mathrm{Ex,Idem}}$, the full subcategory of stable *idempotent-complete* categories, which can equally be characterized as the categories of compact objects of compactly-generated presentable stable categories. Karoubi projections can be characterized as composites of a Verdier projection followed by a dense inclusion (i.e. $\mathcal{C} \to \mathcal{D}$ such that every object of $\mathcal{D}$ is a retract of an object of $\mathcal{C}$) and Karoubi inclusions are simply fully-faithful functors.

On the other hand, *split-Verdier sequences*, also known as *additive sequences* though this is particularly terrible name since all categories in sight are stable and the functors between them exact, and the related *semi-split Verdier sequences*, which are Verdier sequences where the Verdier projection is further asked to have either a left, a right or both adjoints (the first two corresponding to semi-split projections and the latest to the split case). The adjoint of a Verdier projection is automatically a Verdier inclusion, and thus entails the localization $L : \mathcal{C} \to \mathcal{D}$ to be a *Bousfield localization*, i.e. to realize $\mathcal{D}$ as a subcategory of $X \in \mathcal{C}$ such that some morphism (either $X \to L(X)$ or $L(X) \to X$ depending on the side of the adjoint) is an equivalence.

Let us note that the category $\mathbf{Cat}^{\mathrm{Ex}}$ is far from being itself stable; in fact, a result of a further section will show that its stabilization vanishes. It is not even additive since $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})^{\simeq}$ is only a monoid in spaces and not group-like in general. However, it carries some 2-categorical ersatz of an exact structure (even multiple when considering the different flavors of split, non-split Verdier and Karoubi sequences) and trying to understand how much information this captures leads to the next section.

## 2.2 Algebraic K-theory

Due to both its ubiquitous nature and deep ties with homotopical algebra, which is a topic which has known multiple revolutions of language, there is a plethora of presentations of algebraic K-theory. The one we have chosen is probably the most abstract, though not necessarily the most general. It follows along ideas introduced in [BGT13] but we altered the presentation because we will not prove anything.

As we have discussed in the previous section, $\mathbf{Cat}^{\mathrm{Ex}}$ carries many things which are not too far from an exact structure. We also discussed how every exact category admits a stable envelope. It is legitimate to wonder whether such an envelope also still exists, and how it varies from the different structures. Let us say that a functor $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is an *additive invariant* (respectively, Verdier- or Karoubi-localizing) if it sends additive (resp. Verdier, Karoubi) sequences to exact sequences of $\mathcal{E}$. A very general argument (using say 5.2.6.3 of [Lur08]) shows:

**Theorem 2.2 — Blumberg-Gepner-Tabuada.** For the three flavors of invariants cited above, there exists a universal invariant $U : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Mot}_{flavour}$ whose target is stable such that for every stable $\mathcal{E}$:

$$U^* : \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Mot}_{flavour}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}_{flavour}(\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$$

is an equivalence, where the right-hand side is the category of invariants which are of the wanted flavor.

So the answer to our question is at least promising: there is a stable category which, for all intent and purposes, has the correct exact sequences. In upcoming work [RSW25], Ramzi–Sosnilo–Winges show that for the Karoubi-localizing one, the functor $U_{Kar} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{M}_{Kar}$ is actually a localization. In fact, this is also true for the additive one if one is more careful in the formulation: one has to remark that additive invariant should not be defined as having necessarily a stable target but an exact one; in this more general case, the target of the universal functor is a split-exact category whose stabilization is the above $\mathrm{Mot}_{add}$ and only this smaller additive category is a localization. In particular, $\mathrm{Mot}_{add}$ carries a weight-structure. The author will probably write some more precise things regarding this claim at some point in the future.

In $\mathbf{Cat}^{\mathrm{Ex}}$, there is a noteworthy object in the form of $\mathrm{Sp}^{fin}$, the category of finite spectra which is also the free stable category on one object, namely the sphere spectrum $\mathbb{S}$. In fact, $\mathbf{Cat}^{\mathrm{Ex}}$ is generated under colimits by this object, though not freely so.

**Definition 2.3** Let $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ denote the following functor

$$\mathcal{C} \longmapsto \mathrm{map}_{\mathrm{Mot}_{\mathrm{add}}}(U_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}}), U_{\mathrm{add}}(\mathcal{C}))$$

We call this functor the *algebraic K-theory* functor. By construction, K is an additive invariant, and since $\mathrm{Mot}_{\mathrm{add}}$ has a weight structure whose heart is spanned by the image of $U_{\mathrm{add}}$, K lands in connective spectra.

Naturally, this definition looks remarkably obscure for anyone who is familiar with *old-school K-theory*. Part of this thesis is to try and convince the reader that a lot can be achieved through it so we will stick to it until much later, where we will need a concrete construction, and we will use the one described in [Bar13, Bar16]. Let us not say just yet what this concrete construction is, as it is not necessary for most of our exposition.

By construction, mapping spaces in $\mathbf{Cat}^{\mathrm{Ex}}$ are just the underlying groupoids of the functor categories $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})$. Hence $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ comes with a universal property with respect to the functor $\Sigma^\infty_+ \mathrm{Map}_{\mathbf{Cat}^{\mathrm{Ex}}}(\mathrm{Sp}^{\mathrm{fin}}, \mathcal{C})$, i.e. the infinite suspension of the core functor $\iota$, whose proof is recalled in Theorem 4.9. Namely, the natural transformation $\Sigma^\infty_+ \iota \to \mathrm{K}$ is initial among those with target an additive invariant.

Note that since $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)$ and $\mathcal{C} \otimes -$ preserve additive sequences for every stable $\mathcal{C}$, a standard argument (a variation of which is collected in the proof of Proposition 2.4) shows that

$$\mathrm{K}(\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{map}_{\mathrm{Mot}_{\mathrm{add}}}(U_{\mathrm{add}}(\mathcal{C}), U_{\mathrm{add}}(\mathcal{D}))$$

for any $\mathcal{C}, \mathcal{D}$. We warn that this holds because we took the non-presentable version of $\mathrm{Mot}_{\mathrm{add}}$, i.e. the one which factors all additive functors regardless of the cardinal for which they might be accessible; in particular, this is why our statement looks more general than the one presented in [BGT13].

It stands to reason that we can define two other variants of K-theory, $K^{Ver}$ and $\mathbb{K}$ which are respectively initial as Verdier-localizing invariants and Karoubi-localizing invariants. The latter is often known as non-connective K-theory and will slip out of the discussion completely, as the "*correct*" way of doing trace methods for $\mathbb{K}$ certainly lies within the set-up of Efimov in [Efi24] which is out of the scope of this thesis. The former, $K^{Ver}$, is surprisingly equal to K:

**Proposition 2.4** The K-theory functor $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is Verdier-localizing. Moreover, there is an equivalence

$$\mathrm{K}(\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{map}_{\mathrm{Mot}_{\mathrm{Ver}}}(U_{\mathrm{Ver}}(\mathcal{C}), U_{\mathrm{Ver}}(\mathcal{D}))$$

if $\mathcal{C}$ is finite-smooth and finite-proper, where $U_{\mathrm{Ver}} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Mot}_{\mathrm{Ver}}$ is the universal invariant of the Verdier-localizing flavor, as in Theorem 2.2.

**Proof.** The first claim is proven in [Sau23a, Theorem 3.1] or [HLS23, Theorem 6.1] and we refer to the proof there since this section is supposed to be recollection. Note that since it was already initial as an additive invariant Against our better judgment, we provide a proof of the second claim since we could not locate a reference.

We first prove that if $\mathcal{C} \in \mathbf{Cat}^{\mathrm{Ex}}$ is such that $\mathcal{C} \otimes -$ and $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)$ preserve Verdier sequences, then for all $\mathcal{D}$, there is an equivalence:

$$\mathrm{K}(\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{map}_{\mathrm{Mot}_{\mathrm{Ver}}}(U_{\mathrm{Ver}}(\mathcal{C}), U_{\mathrm{Ver}}(\mathcal{D}))$$

The strategy of proof is standard, for instance it features in the proof of Theorem 6.7 in [SW25]. Note that the the right hand side has a universal property as a functor of $\mathcal{D}$, namely

$$\Sigma^\infty_+ \mathrm{map}_{\mathbf{Cat}^{\mathrm{Ex}}}(\mathcal{C}, -) \longrightarrow \mathrm{map}_{\mathrm{Mot}_{\mathrm{Ver}}}(U_{\mathrm{Ver}}(\mathcal{C}), U_{\mathrm{Ver}}(-))$$

is the initial natural transformation with the above source and target a Verdier-localizing invariant.

Since $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)$ preserve Verdier sequences, $\mathrm{K}(\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -))$ is Verdier-localizing. But now, $\mathcal{C} \otimes -$ also preserves Verdier-sequences therefore we have the chain of equivalences

$$\mathrm{Nat}(\mathrm{K}(\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)), F) \simeq \mathrm{Nat}(\mathrm{K}, F(\mathcal{C} \otimes -)) \simeq \mathrm{Nat}(\Sigma^\infty_+ (-)^{\simeq}, F(\mathcal{C} \otimes -)) \simeq \mathrm{Nat}(\Sigma^\infty_+ \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)^{\simeq}, F)$$

for every Verdier-localizing[5] $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$. Thus the computation of mapping spectra in motives.

To conclude, it remains to explain why, if $\mathcal{C}$ is finite-smooth and finite-proper, $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)$ and $\mathcal{C} \otimes -$ preserve Verdier sequences. This is a consequence of the fact that $\mathcal{C}$ is dualizable in $\mathbf{Cat}^{\mathrm{Ex}}$, a fact whose explanation we defer to the next section. □

Here, a stable category is finite-smooth if the spectrally-enriched $\mathrm{map}_{\mathcal{C}}$ is in the image of the Yoneda embedding of $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$, and finite-proper if $\mathrm{map}_{\mathcal{C}}$ lands in $\mathrm{Sp}^{\mathrm{fin}}$. Note that $\mathrm{Sp}^{\mathrm{fin}}$ is idempotent-complete so finite-proper coincides with the usual notion of properness of idempotent-complete stable categories if $\mathcal{C}$ is idempotent-complete.

The second part is almost proven in [BGT13]; they work with $\mathbf{Cat}^{\mathrm{Ex,Idem}}$ instead of $\mathbf{Cat}^{\mathrm{Ex}}$ hence their notion of smoothness and properness are related to compact objects and not finite, and therefore, their computation holds only in Karoubi-localizing motives.

## 2.3 Topological Hochschild homology and its variants

Section 5 is dedicated to THH of stable categories with coefficients, and the first subsection acts as a recollection for the part needed in this section. Still, it will be good to review the more classical side of the story, with as usual, a very modern perspective on it. This is what we set out to do.

Let $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ denote the category of presentable stable categories. There is a symmetric monoidal structure $\otimes$ on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$, often called the *Lurie tensor product*, which is such that $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ is the initial functor which preserves colimits in both variables. Whenever we have a symmetric monoidal category, it makes sense to ask what are its dualizable objects.

**Definition 2.5** An object $X \in \mathcal{C}$ is dualizable with respect to $\otimes$ if there exists a *dual object* $D(X) \in \mathcal{C}$ and maps $\mathrm{coev} : \mathbb{1} \to X \otimes D(X)$, $\mathrm{ev} : X \otimes D(X) \to \mathbb{1}$, where $\mathbb{1}$ is the unit, such that there exists equivalences $(\mathrm{ev} \otimes X) \circ (X \otimes \mathrm{coev}) \simeq \mathrm{id}_X$ and similarly for $D(X)$.

The dual of a dualizable object is unique up to equivalence, and dualizability can be detected in the homotopy category. Hence, it is really a property of an object to be dualizable.

Let us quickly investigates dualizable objects of various categories of categories. We will say that a category $\mathcal{C}$ is *finite-proper* if, for all $X, Y \in \mathcal{C}$, the spectrum $\mathrm{map}(X, Y)$ is a finite spectrum, i.e. in the smallest category $\mathrm{Sp}^{\mathrm{fin}}$ of $\mathrm{Sp}$ closed under finite colimits. It is a non-trivial fact that $\mathrm{Sp}^{\mathrm{fin}}$ is idempotent-complete and therefore finite-proper is equivalent to being simply *proper*, i.e. the $\mathrm{map}(X, Y)$ are *compact* spectra.

We will also say that $\mathcal{C}$ is *finite-smooth* if the functor $\mathrm{Sp}^{\mathrm{fin}} \to \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$ corresponding to the point $\mathrm{map} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{Sp}$ factors through $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}$ via the Yoneda embedding. This time this is not equivalent to being proper, which only asks that map is a compact object of $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$.

**Proposition 2.6** Let $\mathcal{C}$ be a small stable category. Then, $\mathrm{Ind}(\mathcal{C})$ is dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ with dual $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$.

In consequence, a category $\mathcal{C}$ is dualizable in $\mathbf{Cat}^{\mathrm{Ex}}$ if and only if it is (finite-)proper and finite-smooth; an idempotent-complete $\mathcal{C}$ is dualizable in $\mathbf{Cat}^{\mathrm{Ex,Idem}}$ if and only if it is proper and smooth.

**Proof.** The first claim is explained in the discussion preceding Theorem 3.7 in [BGT13] and said Theorem proves the claim about dualizable objects of $\mathbf{Cat}^{\mathrm{Ex,Idem}}$. Let us therefore only explain why dualizable objects of $\mathbf{Cat}^{\mathrm{Ex}}$ are the finite-proper and finite-smooth categories. For any stable $\mathcal{C}$, since $\mathrm{Ind} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ is a monoidal functor, we have two continuous functors

$$\mathrm{coev}_{\mathrm{Ind}\,\mathcal{C}} : \mathrm{Sp} \longrightarrow \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}) \qquad\qquad \mathrm{ev}_{\mathrm{Ind}\,\mathcal{C}} : \mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}) \to \mathrm{Sp}$$

satisfying the wanted triangle identity. Since the dualizability datum is unique, if $\mathcal{C}$ is dualizable, then $\mathrm{Ind}(\mathrm{coev}_{\mathcal{C}}) \simeq \mathrm{coev}_{\mathrm{Ind}\,\mathcal{C}}$ and similarly for the evaluation. From this, it follows that $\mathcal{C}$ is finite-smooth and finite-proper. Reciprocally, those conditions ensure that suitable restrictions factor through the correct objects, so that the dualizability datum of $\mathrm{Ind}(\mathcal{C})$ restricts to one for $\mathcal{C}$. □

[5]Surprisingly, one only need that $F(\mathcal{C} \otimes -)$ is additive in the above proof so we did not use that $\mathcal{C} \otimes -$ preserves Verdier sequences. We still wrote the proof this way because this template of proof generalizes to other situations where the universal invariant does not have a stronger universal property.

Whenever we have a endomorphism of a dualizable object $f : X \to X$, we can take its *trace*: it is the endomorphism of the unit obtained by the composition

$$\mathbb{1} \xrightarrow{\text{coev}} X^\vee \otimes X \xrightarrow{\mathrm{id} \otimes f} X^\vee \otimes X \xrightarrow{\simeq} X \otimes X^\vee \xrightarrow{\text{ev}} \mathbb{1}$$

In $\mathbf{Cat}^{\mathrm{Ex}}$, traces are exact functors $\mathrm{Sp}^{\mathrm{fin}} \to \mathrm{Sp}^{\mathrm{fin}}$, i.e. objects of $\mathrm{Sp}^{\mathrm{fin}}$ and in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$, they are colimit-preserving functors $\mathrm{Sp} \to \mathrm{Sp}$, i.e. objects of Sp.

**Definition 2.7** Let $\mathcal{C}$ be a stable category and $M \in \mathrm{End}^{\mathrm{L}}(\mathrm{Ind}\,\mathcal{C})$. Then, *topological Hochschild homology* of $\mathcal{C}$ with coefficients in $M$ is the spectrum:

$$\mathrm{THH}(\mathcal{C}, M) = \mathrm{tr}(M)$$

where the trace is computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$. Proposition 4.24 of [HSS17] guarantees that this expression coincides with the more usual cyclic Bar construction with many objects of [BGT13, Section 10.1].

**Remark 2.8** If $\mathcal{C}$ is finite-smooth and finite-proper and $M$ actually comes from an exact functor $\mathcal{C} \to \mathcal{C}$, then we could also take the trace in $\mathbf{Cat}^{\mathrm{Ex}}$. This would result in a finite spectrum which coincides with $\mathrm{THH}(\mathcal{C}, M)$, given the discussion above Definition 2.7.

**Warning 2.9** In §5, we will introduce an *a priori* different definition of THH, namely in Definition 5.7. We will prove thereafter that this definition coincides, the exact deduction being recorded in Remark 5.29.

As a trace, THH enjoys the *functoriality of traces*, a somewhat mysterious thing which has been studied by Ponto–Shulman [PS13], Kaledin and Nikolaus [Kal15, Nik18], Ramzi in close relations to trace methods [Ram24a], and which is the main topic of [HNS25]. In essence, it stems from the classical fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for matrices $A, B$, but in our categorified world, this $=$ sign must become a homotopy and it requires some work to set-up the fact that higher cyclic relations yield compatible homotopies. In any case, there is a resulting endowment of $\mathrm{THH}(\mathcal{C}, M^{\otimes n})$ with a action of $\mathrm{C}_n$, the cyclic group with $n$ elements.

Here, $M^{\otimes n}$ is most simply understood as $M$ composed with itself $n$-times when thinking of bimodules as continuous functors $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$. In particular, note that this structure is monoidal but not symmetric, so that this action is not for free.

In fact, THH has more structure: there are functorial, $\mathrm{C}_n$-equivariant maps

$$\phi_{n,p} \colon \mathrm{THH}(\mathcal{C}, M^{\otimes n}) \longrightarrow \mathrm{THH}(\mathcal{C}, M^{\otimes np})^{\mathrm{tC}_p}$$

which form the *polygonic structure* on THH, as in [KMN23]. Here we are getting a little bit ahead of ourselves, since proving those facts in this generality is part of related works, namely [HNS25]. In particular, there is a more refined version of the above where the target is now the *proper* Tate construction, called the *genuine polygonic structure* which we will save for later.

Nonetheless, an offshoot of the polygonic structure has been well-known and studied for years: it is the famed cyclotomic structure on $\mathrm{THH}(\mathcal{C}, \mathrm{id})$, which we will abbreviate $\mathrm{THH}(\mathcal{C})$. What happens is as follows: since $M = \mathrm{id}$, $\mathrm{THH}(\mathcal{C})$ has a wealth of $\mathrm{C}_n$-actions, one for each $n$, which are all compatible under divisibility: the crux of the story is that they actually all come from a $S^1$-action. Furthermore, the maps

$$\phi_p \colon \mathrm{THH}(\mathcal{C}) \longrightarrow \mathrm{THH}(\mathcal{C})^{\mathrm{tC}_p}$$

also are $S^1$-equivariant. Cyclotomic spectra form a category CycSp, introduced in their modern version by Nikolaus-Scholze [NS17] (though this is not always the same as the old-school cyclotomic spectra, as will become painfully relevant in later sections), as the following pullback of presentable stable categories:

$$\begin{tikzcd}
\mathrm{CycSp} \arrow[r] \arrow[d] & \prod_p \mathrm{Ar}(\mathrm{Sp}^{BS^1}) \arrow[d, "{(s,t)}"] \\
\mathrm{Sp}^{BS^1} \arrow[r, "{(\mathrm{id},(-)^{\mathrm{tC}_p})}"] & \prod_p (\mathrm{Sp}^{BS^1} \times \mathrm{Sp}^{BS^1})
\end{tikzcd}$$

There is a colimit-preserving functor $\mathrm{Sp} \to \mathrm{CycSp}$ endowing a spectrum $X$ with the trivial $S^1$-action and structure maps $X \to X^{\mathrm{hC}_p} \to X^{\mathrm{tC}_p}$. The right adjoint of this functor is denoted $\mathrm{TC} : \mathrm{CycSp} \to \mathrm{Sp}$, and precomposing it by THH, we get *topological cyclic homology*, $\mathrm{TC}(\mathcal{C})$, a particularly arduous but computable invariant landing in spectra.

The functor $\mathrm{THH} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{CycSp}$ is Karoubi-localizing. In particular, there is a map $\mathrm{K} \to \mathrm{THH}$ called the *Dennis trace map* which is induced by the choice of 1 in $\mathbb{Z} \simeq \pi_0 \mathrm{THH}(\mathbb{S}) \simeq \pi_0 \mathrm{Nat}(\mathrm{K}, \mathrm{THH})$. This upgrades to a map of cyclotomic spectra when giving K the trivial cyclotomic structure, hence refines to the *cyclotomic trace*, a natural transformation $\mathrm{K} \to \mathrm{TC}$. Note that $\mathrm{TC} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is also localizing, so this cyclotomic trace also corresponds to a point in $\pi_0 \mathrm{TC}(\mathbb{S})$, this latter group is again $\mathbb{Z}$ and we are again taking the map corresponding to 1.

We can now broadly state our goal: we want to investigate the maps

$$\mathrm{K} \longrightarrow \mathrm{TC} \longrightarrow \mathrm{THH}$$

and more precisely, figure out whether the knowledge of one of the sides can provide information on the other two.

# 3 The tangent bundle of $\mathbf{Cat}^{\mathrm{Ex}}$

This section and the two that follow have diverged from an early draft of [HNS24]. In particular, the results there are to be considered joined with Yonatan Harpaz and Thomas Nikolaus, and only the mistakes are of the sole responsibility of the author.

Let us begin by briefly recalling some facts about tangent categories and the tangent bundle of a nice-enough category, which are discussed at length in section 7.3 of [Lur17a]. Let $\mathcal{C}$ be a category with finite limits and denote $\mathcal{S}_*^{\mathrm{fin}}$ the category of finite pointed spaces, the smallest subcategory of $\mathcal{S}_*$ containing $*$ and stable under finite colimits. There is a cocartesian fibration

$$\pi : \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \longrightarrow \mathcal{C}$$

given by the evaluation at the point, where the left-hand-side category is the category of excisive functors $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$. The fiber of $\pi$ over some $X \in \mathcal{C}$ is equivalently given by the category $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_{X//X})$ of *pointed* excisive functors, which is a model for both the stabilization $\mathrm{Sp}(\mathcal{C}_{X//X})$ of the over-undercategory of $X$ and the stabilization of $\mathrm{Sp}(\mathcal{C}_{/X})$.

Following the philosophy of Goodwillie calculus, we think of stable category as the curve-less (we would say flat if not for the usual meaning of the word related to tensor products), linear objects of the higher categorical world and in accordance with that philosophy, we denote $\mathrm{T}_X\mathcal{C} := \mathrm{Sp}(\mathcal{C}_{X//X})$ the *tangent category of* $\mathcal{C}$ *at* $X$. Furthering the analogy with manifolds, the construction we have described is akin to bundling together the tangent categories for varying $X$ hence we call it the *the tangent bundle of* $\mathcal{C}$ and denote it $\mathrm{T}\mathcal{C} := \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$. Evaluation along the map $S^0 \to *$ provides a functor sqz, the square-zero extension functor, given by:

$$\begin{aligned}
\mathrm{sqz} : \mathrm{T}\mathcal{C} &\longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}) \\
(X, M) &\longmapsto (\Omega^\infty_{/X} M \to X)
\end{aligned}$$

which characterizes the tangent bundle as the stable envelope (in the sense of [Lur17a, Section 7.3.1]) of the target projection $t : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$. A classical but seminal example is the tangent bundle of $\mathrm{CAlg}(\mathrm{Sp})$: [Lur17a, Theorem 7.3.4.14] asserts that it has objects pairs $(R, M)$ with

$R \in \mathrm{CAlg}(\mathrm{Sp})$ and $M$ a $R$-bimodule, and sqz is the square-zero extension of ring spectra in the usual sense.

Our goal is to study the tangent bundle of $\mathbf{Cat}^{\mathrm{Ex}}$, the category of stable categories and exact functors between them. Up to Morita equivalences, CAlg(Sp) embeds in $\mathbf{Cat}^{\mathrm{Ex}}$, and the driving idea of our study is that the tangent bundle of $\mathbf{Cat}^{\mathrm{Ex}}$ shares a similar description to the tangent bundle of CAlg(Sp) that we have just recalled.

## 3.1 Bimodules on a stable category, laced categories

Throughout this manuscript, we will call objects of $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$ $\mathcal{C}$-*bimodules*, and often write $\mathrm{Bimod}(\mathcal{C})$ instead of the former. We issue first and foremost the following warning:

**Warning 3.1** A $\mathcal{C}$-bimodule is not a bimodule for a commutative algebra object in $\mathbf{Cat}^{\mathrm{Ex}}$, equipped with its usual tensor product. In fact, the category of such bimodules objects is not even stable.

Having cleared this first hurdle, note that the objects of the category $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ can be described as pairs $(\mathcal{C}, M)$ where $\mathcal{C}$ is stable and $M \in \mathrm{Bimod}(\mathcal{C})$. Since this category will be central in all of this paper and to avoid the cumbersome "stable category equipped with a bimodule $M$ over itself", we have opted to give a name to such pairs:

**Definition 3.2** A *laced category* $(\mathcal{C}, M)$ is the datum of a stable category and a bimodule $M \in \mathrm{Bimod}(\mathcal{C})$. The category of laced categories $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is the tangent bundle of $\mathbf{Cat}^{\mathrm{Ex}}$ and its square-zero extension functor is denoted $\mathrm{Lace} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$.

Our first goal is to get a concrete, tractable expression for $\mathrm{Bimod}(\mathcal{C})$. For this, we shall mostly combine existing results of the literature. By [BGT13, Theorem 1.10], the 1-category $\mathbf{Cat}_{\mathrm{Sp}}$ of spectrally enriched categories can be equipped with a model structure such that the localization at its weak equivalences, which we will denote the same way, admits $\mathbf{Cat}^{\mathrm{Ex}}$ as a full subcategory closed under limits (in fact a left Bousfield localization). In particular, for every stable $\mathcal{C}$, this induces a functor

$$\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \longrightarrow \mathrm{T}_{\mathcal{C}}\mathbf{Cat}_{\mathrm{Sp}}$$

which again identifies its source as a full subcategory closed under limits of its target. Our second ingredient is [HNP19]: indeed Sp can be realized as the category underlying a stable, symmetric monoidal model category which is differentiable and such that the unit $\mathbb{S}$ is compact in the homotopic sense. Hence, it satisfies the hypotheses of [HNP19, Corollary 3.1.17] which gives a natural identification:

$$\mathrm{T}_{\mathcal{C}}\mathbf{Cat}_{\mathrm{Sp}} \simeq \mathrm{Fun}^{\mathrm{Sp}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$$

for any spectrally enriched $\mathcal{C}$. In particular, when $\mathcal{C}$ is stable, the right hand side is exactly the category $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$.

**Proposition 3.3** Let $\mathcal{C}$ be a stable category. The functor

$$\mathrm{Bimod}(\mathcal{C}) \simeq \mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \longrightarrow \mathrm{T}_{\mathcal{C}}\mathbf{Cat}_{\mathrm{Sp}} \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$$

is an equivalence.

**Proof.** It suffices to show that for any spectrally enriched $\mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ is stable, $\Omega_{/\mathcal{C}}\mathcal{D}$ is stable. Assuming this, we deduce a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[rr] \arrow[dr] & & (\mathbf{Cat}_{\mathrm{Sp}})_{/\mathcal{C}} \arrow[rr, "\Omega_{/\mathcal{C}}"'] \arrow[dr] & & (\mathbf{Cat}_{\mathrm{Sp}})_{/\mathcal{C}} \arrow[rr, "\Omega_{/\mathcal{C}}"'] \arrow[dr] & & (\mathbf{Cat}_{Sp})_{/\mathcal{C}} \\
& \cdots \arrow[rr, "\Omega_{/\mathcal{C}}"] \arrow[ur] & & (\mathbf{Cat}^{\mathrm{Ex}})_{/\mathcal{C}} \arrow[rr, "\Omega_{/\mathcal{C}}"] \arrow[ur] & & (\mathbf{Cat}^{\mathrm{Ex}})_{/\mathcal{C}} \arrow[ur] &
\end{tikzcd}$$

This shows that the horizontal towers are equivalent as Pro-objects, in particular the limit of the lower tower is equivalent to the limit of the higher one, but the former is $T_{\mathcal{C}}\mathbf{Cat}_{\mathrm{Sp}}$ while the latter is $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$.

We prove a slightly more general fact than is needed to get the above. Let $\mathcal{A}, \mathcal{B}$ be stable and $\mathcal{C}$ spectrally enriched with two maps $\mathcal{A} \to \mathcal{C}$ and $\mathcal{B} \to \mathcal{C}$ which preserve finite limits and colimits; we show that the fiber product $\mathcal{P} := \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is stable. This will in particular apply when $\mathcal{A} \simeq \mathcal{B}$ is stable and $\mathcal{D}$ is any spectrally enriched category under this common value.

For this, it suffices to see that the loop functor of $\mathcal{P}$ is invertible; by the explicit expression of the mapping spectra of $\mathcal{P}$, we see that $\Omega_{\mathcal{P}}$ has a left adjoint given by $\Sigma_{\mathcal{P}}$. Since $\mathcal{P} \to \mathcal{A} \times \mathcal{B}$ is spectrally-enriched, it will map the unit and the counit to their respective analogues in $\mathcal{A} \times \mathcal{B}$, but this is a stable category so these maps are equivalences and $\mathcal{P} \to \mathcal{A} \times \mathcal{B}$ is conservative. Hence, $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is also stable, which concludes. □

The association $\mathcal{C} \mapsto \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$ admits both a contravariant and a covariant functorial upgrade, by either precomposing or left Kan extending along $f^{\mathrm{op}} \times f$, when $f : \mathcal{C} \to \mathcal{D}$ exact. This yields a bicartesian fibration to $\mathbf{Cat}^{\mathrm{Ex}}$, whose source is none other than $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$.

**Corollary 3.4** The bicartesian fibration $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ classifies the functor

$$\mathcal{C} \longmapsto \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$$

with covariant transition arrows given by left Kan extension and contravariant transition arrows given by precomposition along $(f^{\mathrm{op}} \times f)$ for $f : \mathcal{C} \to \mathcal{D}$ exact.

This identification provides more than just a good handle on objects: it follows that a *laced functor*, i.e. a map $(\mathcal{C}, M) \to (\mathcal{D}, N)$ between laced categories is the datum of an exact functor $f : \mathcal{C} \to \mathcal{D}$ as well as either a natural transformation $\eta : M \to N \circ (f^{\mathrm{op}} \times f)$ or equivalently, $(f^{\mathrm{op}} \times f)_! M \to N$. The following functorial description will also be useful:

**Lemma 3.5** Let $(\mathcal{C}, M)$ and $(\mathcal{D}, N)$ be two laced categories, we have a cartesian square of spaces:

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C}, M), (\mathcal{D}, N)) \arrow[r] \arrow[d] & \iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{D}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})^{\Delta^1} \arrow[d, "{(s,t)}"] \\
\iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \arrow[r] & \iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{D}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})^2
\end{tikzcd}$$

where the bottom vertical map sends $f$ to the couple $((f^{\mathrm{op}} \otimes \mathrm{id})_! M, N \circ (\mathrm{id} \otimes f))$.

**Proof.** Both vertical maps are Kan fibrations, the former by [Lur08, 2.4.4.1], so it suffices to check that the fibers over any point are equivalent, which follows from [Lur08, 2.4.4.2]. □

**Remark 3.6** We have an equivalence $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}))$, and the category of exact presheaves $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ can be identified with $\mathrm{Ind}(\mathcal{C})$, the Ind-completion of $\mathcal{C}$. Using its universal property, we get an equivalence:

$$\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp}) \simeq \mathrm{End}^{\mathrm{L}}(\mathrm{Ind}(\mathcal{C}))$$

In particular, we see that the category $\mathrm{Bimod}(\mathcal{C})$ is invariant under passing to the idempotent completion.

Denote $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ the Yoneda embedding. If $F : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ is a colimit-preserving functor, its image under the above equivalence is $B_F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{Sp}$, given on objects by:

$$B_F(X, Y) := \mathrm{map}_{\mathrm{Ind}(\mathcal{C})}(j(X), F(j(Y)))$$

In the following, we will often omit to write the Yoneda embedding. In particular, with this change of point of view, the above square becomes

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C},M),(\mathcal{D},N)) \arrow[r] \arrow[d] & \iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathrm{Ind}(\mathcal{D}))^{\Delta^1} \arrow[d, "(s,t)"] \\
\iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathcal{D}) \arrow[r] & \iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathrm{Ind}(\mathcal{D}))^2
\end{tikzcd}$$

where the maps sends an exact $f:\mathcal{C}\to\mathcal{D}$ to the pair $(\mathrm{Ind}(f)\circ M, N\circ f)$.

**Remark 3.7** Since the core $\iota$ is a limit-preserving functor, removing the $\iota$ in the previous pullback defines a natural enrichment of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ in $\mathbf{Cat}$, which we will denote $\mathrm{Fun}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}$. Concisely,

$$\mathrm{Fun}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C},M),(\mathcal{D},N)) := \mathrm{LaxEq}\left(\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathcal{D}) \underset{N\circ(-)}{\overset{\mathrm{Ind}(-)\circ M}{\rightrightarrows}} \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C},\mathrm{Ind}\,\mathcal{D})\right)$$

where LaxEq denotes the lax-equalizer construction of [NS17, Definition II.1.4].

■ **Example 3.8** Let $R$ be a ring spectrum, and denote $\mathrm{Perf}(R)$ the category of perfect modules, i.e. compact objects of $\mathrm{Mod}_R$ (our convention will be to consider right-modules in the following). Any non-necessarily perfect $R$-bimodule $M$ determines a bimodule $F_M$ on $\mathrm{Perf}(R)$:

$$F_M : N \in \mathrm{Perf}(R) \mapsto N \otimes_R M \in \mathrm{Mod}_R$$

The association $M \mapsto F_M$ upgrades to a fully-faithful functor $\mathrm{Bimod}_R \to \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Perf}(R),\mathrm{Mod}_R)$. Remark that an exact functor $\mathrm{Perf}(R)\to\mathrm{Mod}_R$ is fully determined by its value at $R$ and $F_M(R)\simeq M$, which acquires the structure of $R$-bimodule via the identification $\mathrm{End}_{\mathrm{Perf}(R)}(R)\simeq R^{\mathrm{op}}$, hence the above construction is in fact an equivalence of categories. ■

As we remarked in 3.7, the enriched mapping categories of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ are lax-equalizers. When plugging as the source $(\mathrm{Sp}^{\mathrm{fin}},\mathrm{id})$ where $\mathrm{id}:\mathrm{Sp}\to\mathrm{Sp}$ is viewed as a colimit-preserving endomorphism of $\mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})$, we get the following object, which will be central in this paper:

**Definition 3.9** Let $M:\mathcal{C}\to\mathrm{Ind}(\mathcal{C})$ be an exact functor. The *category of laces in* $(\mathcal{C},M)$, denoted $\mathrm{Lace}(\mathcal{C},M)$ is the lax-equalizer from the Yoneda embedding of $\mathcal{C}$ to $M$, i.e. the following pullback:

$$\begin{tikzcd}
\mathrm{Lace}(\mathcal{C},M) \arrow[r] \arrow[d] & \mathrm{Ind}(\mathcal{C})^{\Delta^1} \arrow[d] \\
\mathcal{C} \arrow[r, "(j,M)"] & \mathrm{Ind}(\mathcal{C})\times\mathrm{Ind}(\mathcal{C})
\end{tikzcd}$$

This is in particular a stable category by [NS17, II.1.5].

Equivalently, in the bimodule point of view, objects of $\mathrm{Lace}(\mathcal{C},M)$, are described by the data of a point $X\in\mathcal{C}$ as well as a map $\mathbb{S}\to M(X,X)$, or equivalently a point in $\Omega^\infty M(X,X)$, since by Remark 3.6, $\mathrm{map}(-,F(-))$ is the bimodule associated to $F:\mathcal{C}\to\mathrm{Ind}(\mathcal{C})$.

However, note that the arrows between laces in $(\mathcal{C},M)$ and more generally, the whole category $\mathrm{Lace}(\mathcal{C},M)$ is harder to describe: since $M(X,X)$ is not functorial in $X$ we cannot write a similar lax-equalizer between the constant $\mathbb{S}$ and the diagonal $M(X,X)$. Instead, one can realize $\mathrm{Lace}(\mathcal{C},M)$ as the pullback along the diagonal of $\mathcal{C}$ of the bifibration $\mathcal{U}\to\mathcal{C}\times\mathcal{C}$ classifying the functor $\Omega^\infty M:\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{S}$.

■ **Example 3.10** Let $R$ be a ring spectrum and $M$ a bimodule, and denote $F_M$ the associated $\mathrm{Perf}(R)$-bimodule (see Remark 3.8). Then, $\mathrm{Lace}(\mathrm{Perf}(R),F_M)$ is the category of compact modules $N$ equipped with a natural transformation $N\to N\otimes_R M$, which is often called the category of $M$-parametrized endomorphisms.

Suppose further that $R, M$ are connective. Then, $\mathrm{Lace}(\mathrm{Perf}(R),F_{\Sigma M})$ is generated by a single object, namely the pair $(R, 0: R\to\Sigma M)$. Indeed, the connectivity assumption on $M$ implies that all the objects where $N\in\mathrm{Proj}(R)$ are of the form $(N, 0: N\to \Sigma M\otimes_R N)$ hence the full subcategory $\mathrm{Lace}(\mathrm{Proj}(R), F_{\Sigma M})$ obtained by pulling back along $\mathrm{Proj}(R) \to \mathrm{Perf}(R)$ is generated by a single object, namely $S$ the endomorphism ring spectrum of $0 : R \to \Sigma M$.

Consequently, combining Theorem 2.9 and Lemma 3.10[6] of [Sau23b] implies $\mathrm{Lace}(\mathrm{Perf}(R), F_{\Sigma M})$ is equivalent to $\mathrm{Stab}(\mathrm{Proj}(S))$ and $S$ is connective: this is none other than $\mathrm{Perf}(S)$. Computing explicitly this endomorphism ring spectrum via the equalizer formula, we get

$$S \simeq \mathrm{Eq}\left( \mathrm{map}_{\mathrm{Perf}(R)}(R, R) \overset{0}{\underset{0}{\rightrightarrows}} \mathrm{map}_{\mathrm{Perf}(R)}(R, \Sigma M) \right)$$

In particular, the underlying spectrum of $S$ is equivalent to $R \oplus M$. Moreover, under this equivalence, the ring structure identifies it with the square-zero extension $R \oplus M$, hence we recover already at the level of categories the result of Dundas-McCarthy in [DM94], comparing $\mathrm{K}(R \oplus M)$ and $\mathrm{K}(\mathrm{Lace}(\mathrm{Perf}(R), F_{\Sigma M}))$. ■

By construction, we have an equivalence

$$\mathrm{Fun}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}), (\mathcal{C}, M)) \simeq \mathrm{Lace}(\mathcal{C}, M)$$

In particular, we see that Lace is a functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$. Moreover, it has a left adjoint given as follows:

**Proposition 3.11** The functor $\mathrm{Lace} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ is right adjoint to the functor $L$ given on objects by $\mathcal{C} \mapsto (\mathcal{C}, \mathrm{id}_{\mathrm{Ind}\,\mathcal{C}})$.

**Proof.** Let $\mathcal{C}$ be a stable category and $(\mathcal{D}, N)$ a laced category. By Lemma 3.5 and the subsequent remark, $\mathrm{Map}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C}, \mathrm{id}_{\mathrm{Ind}\,\mathcal{C}}), (\mathcal{D}, N))$ is the following pullback square:

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C}, \mathrm{id}_{\mathrm{Ind}\,\mathcal{C}}), (\mathcal{D}, N)) \arrow[r] \arrow[d] & \iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Ind}(\mathcal{D}))^{\Delta^1} \arrow[d, "{(s,t)}"] \\
\iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \arrow[r] & \iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Ind}(\mathcal{D}))^2
\end{tikzcd}$$

where the bottom vertical map sends $f$ to the couple $(\mathrm{Ind}(f) \circ j_{\mathcal{C}}, G \circ f))$ and $j_{\mathcal{C}}$ denotes the Yoneda embedding of $\mathcal{C}$. Since $\mathrm{Ind}(f) \circ j_{\mathcal{C}} \simeq j_{\mathcal{D}} \circ f$, this is the square of Definition 3.9 to which we applied the limit-preserving $\iota\,\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)$, which is the mapping space in $\mathbf{Cat}^{\mathrm{Ex}}$. In particular, we deduce from this a natural equivalence

$$\mathrm{Map}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C}, \mathrm{id}), (\mathcal{D}, N)) \simeq \mathrm{Map}_{\mathbf{Cat}^{\mathrm{Ex}}}(\mathcal{C}, \mathrm{Lace}(\mathcal{D}, N))$$

which gives the wanted adjunction. □

By [HNP19, Corollary 1.0.2], the functor $\mathcal{C} \mapsto (\mathcal{C}, \mathrm{id}_{\mathrm{Ind}\,\mathcal{C}})$, which we can equivalently describe as $\mathcal{C} \mapsto (\mathcal{C}, \mathrm{map}_{\mathcal{C}})$, is the cotangent complex of $\mathbf{Cat}^{\mathrm{Ex}}$, up to a suspension which we can absorb in the abstract equivalence $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$. Consequently, we have the following:

**Proposition 3.12** The functor $\mathrm{Lace} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ coincides with the square-zero extension of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. In particular, on each fiber, postcomposition by the restriction $\mathrm{Lace}(\mathcal{C}, -)$ induces an equivalence, natural in stable $\mathcal{D}$:

$$\mathrm{Lace}(\mathcal{C}, -)_* : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{D}, \mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{REx}}(\mathcal{D}, (\mathbf{Cat}^{\mathrm{Ex}})_{\mathcal{C}//\mathcal{C}})$$

Moreover, $\mathrm{Lace}(\mathcal{C}, -) : \mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \to (\mathbf{Cat}^{\mathrm{Ex}})_{/\mathcal{C}}$ has a left adjoint given pointwise by $(f : \mathcal{D} \to \mathcal{C}) \mapsto (\mathcal{C}, (f^{\mathrm{op}} \times f)_! \,\mathrm{map}_{\mathcal{D}})$.

**Proof.** All of the proposition follows immediately from the identification of [HNP19, Corollary 1.0.2] of the cotangent complex of $\mathbf{Cat}_{\mathrm{Sp}}$ and Proposition 3.11, save for the description of the left adjoint of $\mathrm{Lace}(\mathcal{C}, -)$. Since $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ is a bicartesian fibration, this left adjoint is given pointwise at $f : \mathcal{D} \to \mathcal{C}$ by the target of the cocartesian arrow over $f$ with source $(\mathcal{D}, \mathrm{map}_{\mathcal{D}})$. The explicit description of Proposition 3.11 implies that the cocartesian transfer maps over some $f : \mathcal{D} \to \mathcal{C}$ is given by left Kan extension along $f^{\mathrm{op}} \times f$, which concludes. □

We conclude this section, by the following fact, lifted *verbatim* from [Sau23b, Lemma 3.7]:

[6]These results are also presented almost *verbatim* in §10, see respectively Theorem 10.10 and Lemma 10.21.

**Lemma 3.13 — Lemma 3.7 of [Sau23b].** Let $M$ be a $\mathcal{C}$-bimodule. The canonical functor

$$\iota\,\mathrm{Lace}(\mathcal{C}, M) \to \iota\,\mathcal{C}$$

is the unstraightening of the functor $\iota\,\mathcal{C} \to \mathcal{S}$ sending $X$ to $\mathrm{Map}(X, M(X))$, where $\iota$ denotes the core of a category.

**Proof.** Consider $\mathcal{P}$ the category given by the pullback

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathrm{TwAr}(\mathrm{Ind}\,\mathcal{C}) \arrow[d] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "(j^{\mathrm{op}}, M)"] & \mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Ind}\,\mathcal{C} \end{tikzcd}$$

For a general category $\mathcal{D}$, $\mathrm{TwAr}(\mathcal{D}) \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ classifies $\mathrm{Map}_{\mathcal{D}}(-,-)$, hence it follows that $\mathcal{P} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ classifies the functor $(X, Y) \mapsto \mathrm{Map}(X, M(Y))$. Then, the lemma follows from the following pullback square:

$$\begin{tikzcd} \iota\,\mathrm{Lace}(\mathcal{C}, M) \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\ \iota\,\mathcal{C} \arrow[r, "\Delta"] & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \end{tikzcd}$$

where $\Delta : \iota\,\mathcal{C} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is the diagonal $X \mapsto (X, X)$, using the canonical identification $\iota\,\mathcal{C} \simeq \iota(\mathcal{C}^{\mathrm{op}})$. By pasting, the above square is cartesian as soon as the following square also is:

$$\begin{tikzcd} \iota\,\mathrm{Lace}(\mathcal{C}, M) \arrow[r] \arrow[d] & \mathrm{TwAr}(\mathrm{Ind}\,\mathcal{C}) \arrow[d] \\ \iota\,\mathcal{C} \arrow[r] & \mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Ind}\,\mathcal{C} \end{tikzcd}$$

By the explicit description of pullbacks in $\mathbf{Cat}^{\mathrm{Ex}}$, we can replace the right vertical map by $\iota\,\mathrm{TwAr}(\mathrm{Ind}\,\mathcal{C}) \to \iota\,\mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \times \iota\,\mathrm{Ind}\,\mathcal{C}$. Now, the claim follows from the fact that $\iota$ preserves pullbacks, that $\iota\,\mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \simeq \iota\,\mathrm{Ind}(\mathcal{C})$ and $\iota\,\mathrm{TwAr}(\mathrm{Ind}\,\mathcal{C}) \simeq \iota\,\mathrm{Ar}(\mathrm{Ind}\,\mathcal{C})$. $\square$

## 3.2 Global limits and colimits in the tangent bundle

The functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ is a bicartesian fibration with backwards transition maps given by restriction and forward transition maps given by left Kan extensions. This is a similar setting to [CDH+23b, Section 1.1.4] and by the same methods, we will be able to show that $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ admits all limits and colimits, and explain how they are computed.

Note that there are two kinds of colimits of interest in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$: those internal to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, and the "fiberwise colimits", by which we mean the colimits computed in some stable subcategory $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$, i.e. colimits only in the bimodule entry. The latter will play a critical role in further sections but for now, we focus on the first.

Recall that $\mathbf{Cat}^{\mathrm{Ex}}$ admits all limits and colimits by [CDH+23a, 6.1.1], hence combining 4.3.1.11 and 4.3.1.5.(2) in [Lur08], we get:

**Proposition 3.14** The category $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ has all small limits and colimits and the canonical functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ preserves both.

The above indicates the strategy to compute a colimit in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. First, compute the colimit of the underlying diagram of stable category, and denote $\hat{\mathcal{C}}$ the result; then left Kan extend[7] every bimodule so that they become bimodules over $\hat{\mathcal{C}}$. This yields a diagram in $T_{\hat{\mathcal{C}}}\mathbf{Cat}^{\mathrm{Ex}}$ whose colimit $\hat{M}$ yields a laced category $(\hat{\mathcal{C}}, \hat{M})$ which is the wanted colimit.

We deduce from this a recognition principle for fiber and cofiber sequences of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. First, let us introduce the following terminology:

[7]To compute a limit, the process is the same but one would have to restrict instead of left Kan extending.

**Definition 3.15** A map $(f,\eta):(\mathcal{C},F)\to(\mathcal{D},G)$ is called *left-strict* if $\eta : F\to (f\otimes f^{\mathrm{op}})^*G$ is an equivalence, i.e. the restriction of $G:\mathcal{D}^{\mathrm{op}}\otimes\mathcal{D}\to \mathrm{Sp}$ along $f\otimes f^{\mathrm{op}}$ is $F$, and *right-strict* if $\eta : F\to (f\otimes f^{\mathrm{op}})^*G$ realizes[a] the left Kan extension of $F$ along $f\otimes f^{\mathrm{op}}$.

[a] By this, we mean that the $\eta$ is the natural transformation $F\to (f\otimes f^{\mathrm{op}})^*((f\otimes f^{\mathrm{op}})_!F)$ given as part of the data of the left Kan extension $(f\otimes f^{\mathrm{op}})_!F$.

We remark that left-strict maps are exactly the cartesian transition arrows for the bicartesian fibration $\mathrm{fgt}:\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}\to\mathbf{Cat}^{\mathrm{Ex}}$ and right-strict the cocartesian transitions arrows.

**Proposition 3.16** Let $e : (\mathcal{C},F)\xrightarrow{(i,\alpha)}(\mathcal{D},G)\xrightarrow{(p,\beta)}(\mathcal{E},H)$ be a sequence with nullcomposite. Denote $e^{st}$ the underlying sequence of stable categories. Then,

- $e$ is a fiber sequence in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ if and only if $e^{st}$ is a fiber sequence and $(i,\alpha)$ is left-strict.
- $e$ is a cofiber sequence in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ if and only if $e^{st}$ is a cofiber sequence and $(p,\beta)$ is right-strict.

**Proof.** This is an application of [Lur17a, 7.3.1.12]. □

**Remark 3.17** Definition 3.15 used the description of bimodules as exact functors $\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}\to\mathrm{Sp}$, but it is also convenient to have a description for colimit preserving endofunctors of $\mathrm{Ind}(\mathcal{C})$, then, $(f,\eta):(\mathcal{C},F)\to(\mathcal{D},G)$ is left strict if the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}) \arrow[r, "\mathrm{Ind}(f)"] \arrow[d, "F"] & \mathrm{Ind}(\mathcal{D}) \arrow[d, "G"] \\
\mathrm{Ind}(\mathcal{C}) & \mathrm{Ind}(\mathcal{D}) \arrow[l, "\mathrm{Ind}(f)^r"]
\end{tikzcd}$$

The reversal for the arrow $\mathrm{Ind}(f)^r$ is owed to the fact that $\mathrm{Ind}(f)$ is obtained as left Kan extension $(f^{\mathrm{op}})_!$ when seeing $\mathrm{Ind}(\mathcal{C})=\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})$, whose right adjoint is composition by $f^{\mathrm{op}}$, and this is what we are interested in[a]. Conversely, $(f,\eta):(\mathcal{C},F)\to(\mathcal{D},G)$ is right-strict if the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}) \arrow[d, "F"] & \mathrm{Ind}(\mathcal{D}) \arrow[l, "\mathrm{Ind}(f)^r"'] \arrow[d, "G"] \\
\mathrm{Ind}(\mathcal{C}) \arrow[r, "\mathrm{Ind}(f)"'] & \mathrm{Ind}(\mathcal{D})
\end{tikzcd}$$

Remark that either condition is neither that the natural transformation of squares we are usually writing for endofunctors of $\mathrm{Ind}(\mathcal{C})$ is an equivalence, nor that its Beck-Chevalley (push-pull) associated transformation is an equivalence. When the underlying sequence of stable categories is Karoubi, then the respective left-strict and right-strict condition are implied by the stronger counterparts stated above, but for symmetry reasons, this has no chance of holding in general.

[a] We find this is easiest understood when writing $F(X,Y)=G(f(X),f(Y))$ and then trying to figure what $G(-,f(Y))$ should equal. One needs that $F(-,Y)$ is the precomposition by $(f^{\mathrm{op}})^*$ of $G(-,f(Y))$, hence the commutative square.

Let us conclude this section by showing that $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is in fact presentable and generated under colimits by rather few objects.

**Theorem 3.18** The category $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is presentable, and in fact is generated under colimits by only two objects: $(\mathrm{Sp}^{\mathrm{fin}},0)$ and $(\mathrm{Sp}^{\mathrm{fin}},\mathrm{id})$, which are further compact.

**Proof.** It is clear that $(\mathrm{Sp}^{\mathrm{fin}},0)$ is compact since $\mathrm{Sp}^{\mathrm{fin}}$ is compact in $\mathbf{Cat}^{\mathrm{Ex}}$ and $\mathrm{fgt}:\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}\to\mathbf{Cat}^{\mathrm{Ex}}$, which corepresents it, is colimit-preserving. We also claim that $\iota\,\mathrm{Lace}$ commutes with filtered colimits so that $(\mathrm{Sp}^{\mathrm{fin}},\mathrm{id})$ is also compact. Indeed, recall from Lemma 3.13 that there is a pullback square of categories

$$\begin{tikzcd}
\mathrm{Lace}(\mathcal{C}, M) \arrow[r] \arrow[d] & \int_{Y \in \mathcal{C}}^{X \in \mathcal{C}} \Omega^\infty M(X,Y) \arrow[d] \\
\mathcal{C} \arrow[r, "\Delta"] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

Since pullbacks of categories commute with filtered colimits, it suffices to show the span commutes with filtered colimits. For the bottom map, this is another instance of the previous phenomenon, since products are pullbacks over $*$. For the vertical map, we note that it is equivalent to prove that the functor

$$\Omega^\infty M : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{S}$$

classified by the vertical map is stable under filtered colimits of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. But there is a commutative diagram

$$\begin{tikzcd}
\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \arrow[r] \arrow[d] & \int^{\mathcal{A} \in \mathbf{Cat}} \mathrm{Fun}(\mathcal{A}^{\mathrm{op}} \times \mathcal{A}, \mathcal{S}) \arrow[d] \\
\mathbf{Cat}^{\mathrm{Ex}} \arrow[r] & \mathbf{Cat}
\end{tikzcd}$$

where the bottom map $\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}$ preserves filtered colimits. Note that the functor on fibers which postcomposes by the finitary $\Omega^\infty$ and precomposes by $\iota\mathcal{C} \times \iota\mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, is filtered-colimit preserving; further, the top map is a map of cocartesian fibrations (i.e. preserves cocartesian lifts) so that, the resulting functor on total spaces

$$\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \int^{\mathcal{A} \in \mathbf{Cat}} \mathrm{Fun}(\mathcal{A}^{\mathrm{op}} \times \mathcal{A}, \mathcal{S})$$

also preserves filtered colimits, which is what we wanted.

We recall that a cocomplete category $\mathcal{C}$ is generated under colimits by $X_i$ if $\mathrm{Map}(X_i, -) : \mathcal{C} \to \mathcal{S}$ jointly detect equivalences by [Yan21, Corollary 2.5] and the following remark (this also features in section 2 of [CDH$^+$23d]) and if the $X_i$ form a set, then the category is presentable.

It is a folklore result that $\mathbf{Cat}^{\mathrm{Ex}}$ is compactly generated by $\mathrm{Sp}^{\mathrm{fin}}$. This can be found in [KNP24], and Efimov attributes it to Lurie in the following Lecture. We only record the central argument: fix $f : \mathcal{C} \to \mathcal{D}$ an exact functor which induces an equivalence on cores. Then $f$ is in particular essentially surjective. If $\alpha, \beta : X \to Y$ are equalized by $f$, then their equalizer vanishes in $\mathcal{D}$ i.e. receives an equivalence from 0 which we can lift back to $\mathcal{C}$; since $f$ is an equivalence on spaces of equivalences, the equalizer must vanish before taking $f$ thus $\alpha \simeq \beta$. Now remark that every nilpotent arrow $\gamma : f(X) \to f(X)$ is in the image of $f$: indeed, $\mathrm{id} - \gamma$ is an equivalence with inverse the sum of powers of $\gamma$ so lifts back to some map $\delta$ and $\mathrm{id} - \delta$ must map to $\gamma$ under $f$. Now the conclusion follows from writing every map $\gamma : f(X) \to f(Y)$ as:

$$f(X) \xrightarrow{i_X} f(X) \oplus f(Y) \xrightarrow{\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}} f(X) \oplus f(Y) \xrightarrow{p_Y} f(Y)$$

Indeed, the middle map is nilpotent and the two extremal maps must be in the image of $f$ since $f$ is additive. Upgrading those arguments to more homotopical considerations yields the fact that $f$ is fully-faithful.

By Lemma 3.5, the fiber at $X$ of

$$\mathrm{Fun}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}), (\mathcal{C}, M)) \simeq \mathrm{Lace}(\mathcal{C}, M) \longrightarrow \mathcal{C} \simeq \mathrm{Fun}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathrm{Sp}^{\mathrm{fin}}, 0), (\mathcal{C}, M))$$

is precisely the space of maps $X \to M(X)$. Therefore, given a map $(f, \eta) : (\mathcal{C}, M) \to (\mathcal{D}, N)$ such that $\mathrm{Lace}(f, \eta)$ and $f$ induce equivalences of cores, then both $f$ and $\mathrm{Lace}(f, \eta)$ are equivalences of stable categories by the above argument and this implies that $\eta : M \to N \circ (f^{\mathrm{op}} \times f)$ induces an equivalence on the diagonal. Since $M(X,Y)$ is always a retract of $M(X \oplus Y, X \oplus Y)$, this implies that $\eta$ is always an equivalence, which concludes. □

### 3.3 The symmetric monoidal structure of the tangent bundle

The previous section began by recalling the symmetric monoidal structure on $\mathbf{Cat}^{\mathrm{Ex}}$. We now want to promote this structure to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, in a way that makes the forgetful functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ monoidal. This follows from the formula

$$\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \simeq \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathbf{Cat}^{\mathrm{Ex}})$$

using Day convolution. Let us also write something more explicit:

**Definition 3.19** Let $F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{Sp}$ and $G : \mathcal{D}^{\mathrm{op}} \otimes \mathcal{D} \to \mathrm{Sp}$ be two bimodules. Then the biexact

$$(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \times (\mathcal{D}^{\mathrm{op}} \otimes \mathcal{D}) \xrightarrow{F \times G} \mathrm{Sp} \times \mathrm{Sp} \xrightarrow{\wedge} \mathrm{Sp}$$

refines canonically to an exact functor $(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \otimes (\mathcal{D}^{\mathrm{op}} \otimes \mathcal{D}) \to \mathrm{Sp}$, i.e. a $\mathcal{C} \otimes \mathcal{D}$-bimodule which we denote $F \boxtimes G$.

**Proposition 3.20** The association $((\mathcal{C}, F), (\mathcal{D}, G)) \mapsto (\mathcal{C} \otimes \mathcal{D}, F \boxtimes G)$ refines canonically to a functor which we denote $\otimes : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \times \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. It endows $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ with a symmetric monoidal structure such that $\mathrm{fgt} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ is symmetric monoidal.

Moreover, the unit of the above described symmetric monoidal structure $(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \otimes)$ is given by $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$, which is the category of finite spectra $\mathrm{Sp}^{\mathrm{fin}}$ equipped with $\mathrm{id}_{\mathrm{Sp}}$ as a bimodule.

**Proof.** Using the identification $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \simeq \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathbf{Cat}^{\mathrm{Ex}})$, the forgetful functor fgt is given by restriction along $* \to \mathcal{S}_*^{\mathrm{fin}}$ given by the point. We can then simply apply the results of [Lur17a, Section 2.2.6] (in fact, we only need what Glasman does in [Gla16]). $\square$

Note that there is a canonical identification $\mathrm{map}_{\mathcal{C} \otimes \mathcal{D}} \simeq \mathrm{map}_{\mathcal{C}} \otimes \mathrm{map}_{\mathcal{D}}$, therefore:

**Corollary 3.21** The cotangent complex $L : \mathcal{C} \mapsto (\mathcal{C}, \mathrm{id})$ admits a canonical enhancement as a monoidal functor. Therefore its right adjoint Lace is canonically endowed with a lax monoidal structure functor.

By Remark 3.7, the unit $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}_{\mathrm{Sp}})$ of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ corepresents the functor $\mathrm{Lace}^{\simeq} := \iota\,\mathrm{Lace}$ where $\iota$ denotes the core-groupoid functor $\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{S}$. We can thus extract from results of [Nik16] another universal property for $\mathrm{Lace}^{\simeq}$:

**Lemma 3.22** The functor $\mathrm{Lace}^{\simeq}$ refines to a lax-monoidal functor and this refinement is in fact the initial lax-monoidal functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{S}$. This refinement makes $\Sigma^\infty_+ \mathrm{Lace}^{\simeq}$ into the initial lax-monoidal functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$.

**Proof.** By Proposition 3.20, $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is symmetric monoidal with unit $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}_{\mathrm{Sp}})$, hence the first part follows from [Nik16, Corollary 6.8]. The second part follows as does point (4) of Corollary 6.9 of *loc. cit.*: post-composition by $\Sigma^\infty_+$ is monoidal, hence induces a functor

$$\mathrm{Fun}_{\mathrm{lax}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{S}) \longrightarrow \mathrm{Fun}_{\mathrm{lax}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{Sp})$$

which preserves colimits hence initial objects. $\square$

**Remark 3.23** A priori, we also have at our disposal another type of tensor product: since an exact $F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathrm{Sp}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule, for any exact $G : \mathcal{D}^{\mathrm{op}} \otimes \mathcal{E} \to \mathrm{Sp}$, there is a functor $F \otimes_{\mathcal{D}} G : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{E} \to \mathrm{Sp}$ which is given by the tensor product of bimodules (as in section 4.4 of [Lur17a]).

When $\mathcal{C} = \mathcal{D} = \mathcal{E}$, both tensors are well-defined but are clearly different, as what we just described gives a $\mathcal{C}$-bimodule whereas the construction of Definition 3.20 yields a $(\mathcal{C} \otimes \mathcal{C})$-bimodule.

We have obtained a symmetric monoidal structure on $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ whose unit is $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$. Note that the tensor product we have produced commutes with colimits in both variables; Theorem

3.18 thus implies the symmetric monoidal structure is closed. In fact, we will further show it is both tensored and cotensored over the category of categories $\mathcal{C}$ equipped with a given map $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. First, let us give a more explicit description of the internal mapping objects:

**Definition 3.24** Fix $(\mathcal{C}, F), (\mathcal{D}, G)$ two laced categories and let $f, g : \mathcal{C} \to \mathcal{D}$ be two exact functors. We define the *spectrum of* $(F, G)$*-linear natural transformations* from $f$ to $g$ by the formula:

$$\mathrm{Nat}^F_G(f, g) := \mathrm{Nat}(F, (f^{\mathrm{op}} \otimes g)^* G)$$

where the right hand side object is the spectrum given by the enrichment in Sp of the stable category $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$. This is exact in both $f$ and $g$, covariant in $g$ and contravariant in $f$, hence defines a bimodule $\mathrm{Nat}^F_G$ on $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})$. We denote $\underline{\mathrm{Fun}}((\mathcal{C}, F), (\mathcal{D}, G))$ the associated laced category.

**Remark 3.25** Remark that a map $(f, \alpha) : (\mathcal{C}, F) \to (\mathcal{D}, G)$ in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is given by an exact functor $f : \mathcal{C} \to \mathcal{D}$, i.e. an object of $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})$ and then an object of $\mathrm{Nat}^F_G(f, f)$ (or rather of its associated infinite loop space), hence is encapsulated in the data of $\underline{\mathrm{Fun}}((\mathcal{C}, F), (\mathcal{D}, G))$.

The category $\mathbf{Cat}^{\mathrm{Ex}}$ is closed and thus there is an exact, natural evaluation functor $\mathrm{ev} : \mathcal{C} \otimes \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$. In the presence of bimodules $F, G$, we claim this refines to a map of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, whose underlying exact functor is the evaluation functor:

$$(\mathrm{ev}, \eta_{\mathrm{ev}}) : (\mathcal{C}, F) \otimes \underline{\mathrm{Fun}}((\mathcal{C}, F), (\mathcal{D}, G)) \longrightarrow (\mathcal{D}, G)$$

To provide $\eta_{\mathrm{ev}} : F \boxtimes \mathrm{Nat}^F_G \implies (\mathrm{ev}^{\mathrm{op}} \times \mathrm{ev})^* G$, first consider the evaluation

$$\hat{\mathrm{ev}} : F \wedge \mathrm{Nat}(F, -) \implies \mathrm{id}$$

of $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$ where $\wedge$ denotes the smash product of Sp, applied pointwise at the target.

We claim that precomposition by $(\mathrm{ev}^{\mathrm{op}} \times \mathrm{ev})^* G$ refines to the correct transformation. Indeed, this is a game of currying: $(\mathrm{ev}^{\mathrm{op}} \times \mathrm{ev})^* G$ is equivalently a functor

$$G((-), (-)) : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \times \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{D}^{\mathrm{op}} \otimes \mathcal{D}, \mathrm{Sp})$$

where the precomposition of $\hat{\mathrm{ev}}$ makes sense. Then, currying back induces a natural transformation $F \boxtimes \mathrm{Nat}^F_G \implies (\mathrm{ev}^{\mathrm{op}} \times \mathrm{ev})^* G$ by construction of $\boxtimes$.

**Proposition 3.26** Let $(\mathcal{C}, F) \in \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. The functor $(\mathcal{C}, F) \otimes - : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is left adjoint to $\underline{\mathrm{Fun}}((\mathcal{C}, F), -)$, with counit given by the aforementioned $(\mathrm{ev}, \eta_{\mathrm{ev}})$.

**Proof.** On the underlying stable categories, since $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)$ is indeed right adjoint to $\mathcal{C} \otimes -$ in $\mathbf{Cat}^{\mathrm{Ex}}$ via the evaluation, the counit map induces an equivalence. Consequently, since $\mathrm{fgt} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ is a bicartesian fibration, we have a commutative square whose vertical legs are right fibrations:

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C}, F), \underline{\mathrm{Fun}}((\mathcal{D}, G), (\mathcal{E}, H))) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C}, F) \otimes (\mathcal{D}, G), (\mathcal{E}, H)) \arrow[d] \\
\iota\, \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{D}, \mathcal{E})) \arrow[r, "\simeq"] & \iota\, \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})
\end{tikzcd}$$

Hence, it remains to show that the fibers of the vertical arrows are equivalent. Fixing $\phi : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{D}, \mathcal{E})$, or equivalently $\overline{\phi} : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ and using the description of the fibers provided by Lemma 3.5, it suffices to see that the map induced by $\eta_{\mathrm{ev}}$ is an equivalence:

$$\mathrm{Nat}(F, (\phi^{\mathrm{op}} \times \phi)^* \, \mathrm{Nat}^G_H) \xrightarrow{\simeq} \mathrm{Nat}(F \boxtimes G, (\overline{\phi}^{\mathrm{op}} \times \overline{\phi})^* H)$$

The evaluation $\hat{\mathrm{ev}}$ of $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$ is the counit of the adjunction between $F \wedge -$ and $\mathrm{Nat}(F, -)$. Hence, we have equivalences natural in $X \in \mathrm{Sp}$ and $\hat{F} \in \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$

$$\mathrm{Nat}(F \wedge X, \hat{F}) \simeq \mathrm{Map}(X, \mathrm{Nat}(F, \hat{F}))$$

Precomposing by $(\mathrm{ev}^{\mathrm{op}} \times \mathrm{ev})^* G$ the $X$-variable and $(\overline{\phi}^{\mathrm{op}} \times \overline{\phi})^* H$ the $\hat{F}$-variable, we get the wanted equivalence up to some currying on one side, and since this is how we have built $\eta_{\mathrm{ev}}$, this is indeed the wanted map. $\square$

**Remark 3.27** Since $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}_{\mathrm{Sp}})$ is the unit of the tensor product of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, we get a natural equivalence

$$\underline{\mathrm{Fun}}((\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}_{\mathrm{Sp}}), (\mathcal{C}, F)) \simeq (\mathcal{C}, F)$$

Moreover, we see that

$$\mathrm{Lace}\,\underline{\mathrm{Fun}}((\mathcal{C}, F), (\mathcal{D}, F)) \simeq \mathrm{Fun}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C}, F), (\mathcal{D}, F))$$

since $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}_{\mathrm{Sp}})$ corepresents Lace in $\mathbf{Cat}^{\mathrm{Ex}}$. This relates this enrichment with the one described in Remark 3.7.

**Remark 3.28** Proposition 3.11 of [Nik16] also gives a apriori formula for the internal mapping object of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. In particular, the above Proposition can be deduced from the computation of ends in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$: as any limit in a cartesian unstraightening, one first compute the end of the underlying object in $\mathbf{Cat}^{\mathrm{Ex}}$ which must be the usual enrichment of $\mathbf{Cat}^{\mathrm{Ex}}$ over itself given by $\mathrm{Fun}^{\mathrm{Ex}}(-,-)$ and then pulling along each cartesian transition functor, one computes the resulting end in the fiber, and this time we find the end in the category of functors $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathrm{Sp})$ of the pulled functors, which we have called $\mathrm{Nat}_G^F$ in Definition 3.24.

## 3.4 Tensors and cotensors in the tangent bundle

If $I$ is any category, then there is a functor $I \mapsto \mathrm{Bimod}^{un}(I) := \mathrm{Fun}(I^{\mathrm{op}} \times I, \mathcal{S})$ with a covariant functoriality given by left Kan extension and contravariant functoriality given by restriction. When $I$ is stable, there is a natural transformation $\mathrm{Bimod}^{un}(I) \to \mathrm{Bimod}(I)$ to the above category by forgetting the exactness of the functor and postcomposing by $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$.

In consequence, there is a bicartesian fibration $\mathbf{Cat}^b \to \mathbf{Cat}$ classifying the above functor and it induces a functor $U : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^b$ which preserves both cartesian and cocartesian lifts over the inclusion $\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}$. Remark that $\mathbf{Cat}^b$ is enriched[8] in $\mathbf{Cat}$, hence we can make the following two definitions:

**Definition 3.29** Let $(I, F) \in \mathbf{Cat}^b$ and $(\mathcal{C}, M) \in \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, we denote $(\mathcal{C}, M)^{(I,F)}$ the *cotensor* of $(\mathcal{C}, M)$ and $(I, F)$, characterized by the following universal property:

$$\mathrm{Fun}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{D}, N), (\mathcal{C}, M)^{(I,F)}) \simeq \mathrm{Fun}_{\mathbf{Cat}^b}((I, F), \underline{\mathrm{Fun}}((\mathcal{D}, N), (\mathcal{C}, M)))$$

Dually, we denote $(\mathcal{C}, M)_{(I,F)}$ the *tensor* of $(\mathcal{C}, M)$ and $(I, F)$, characterized by the following universal property:

$$\mathrm{Fun}_{\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}}((\mathcal{C}, M)_{(I,F)}, (\mathcal{D}, N)) \simeq \mathrm{Fun}_{\mathbf{Cat}^b}((I, F), \underline{\mathrm{Fun}}((\mathcal{C}, M), (\mathcal{D}, N)))$$

**Proposition 3.30** For every $(I, F) \in \mathbf{Cat}^b$ and $(\mathcal{C}, M) \in \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, the object $(\mathcal{C}, M)^{(I,F)}$ exists and can be taken to be the stable category $\mathrm{Fun}(I, \mathcal{C})$ equipped with the bimodule $M^{(I,F)}$ given by the following formula:

$$M^{(I,F)}(f, g) := \mathrm{Nat}(F, (f^{\mathrm{op}} \times g)_* M)$$

**Proof.** This is essentially the same proof as Proposition 3.26 which was done when $I$ was stable and $F$ was further assumed exact in both variable (in the proof, $I$ is denoted $\mathcal{C}$); those assumptions were only used to guarantee that we could restrict to categories of exact functors $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, -)$, as we have dropped this requirement here, the proof goes through *mutatis mutandis*. $\square$

Suppose $F, G : \mathcal{A} \to \mathcal{S}$ are two functors valued in spaces and denote $p : \mathrm{Un}(F) \to \mathcal{A}^{\mathrm{op}}$ the left

[8] In fact, we could redo a lot of the previous section with $\mathbf{Cat}^b$ instead and show it is presentable, symmetric monoidal. Note however that it is *not* the tangent of $\mathbf{Cat}$; indeed, it is an even further refinement of it, the unstraightening of $I \mapsto \mathrm{Fun}(\mathrm{TwAr}(I), \mathcal{S})$ which is.

fibration classifying $F$. Then $F$ is the left Kan extension of the functor $\mathrm{cst}_* : \mathrm{Un}(F) \to \mathcal{S}$ constant with value $*$ along $p$ by [GHN17, Corollary 7.5]. Hence, we get that

$$\mathrm{Nat}(F, G) \simeq \mathrm{Nat}(\mathrm{cst}_*, G \circ p^{\mathrm{op}}) \simeq \lim_{\mathrm{Un}(F)} G \circ p^{\mathrm{op}}$$

This allows use to compute the bimodule in some examples where $p$ is well understood.

■ **Example 3.31** Suppose $F : I^{\mathrm{op}} \times I \to \mathcal{S}$ is constant at the point $*$, then $F$ is classified by the identity $\mathrm{id}_{I^{\mathrm{op}} \times I}$. Let $(\mathcal{C}, M)$ be a laced category, then:

$$\mathrm{Nat}(*, (f^{\mathrm{op}} \times g)^* M) \simeq \lim_{(i,j) \in I^{\mathrm{op}} \times I} M(f(i), g(j))$$

In particular, in the case where $I = [n]$ is the linearly order poset with $n$ elements, we get that

$$M^{([n],*)}(f, g) \simeq M(f(n), g(0))$$

Viewing a functor $f : [n] \to \mathcal{C}$ as the data of objects $X_0, ..., X_n$ with maps $X_i \to X_{i+1}$, the above bimodule is simply $M(X_n, Y_0)$ for two chains $(X_i)$ and $(Y_i)$. ■

■ **Example 3.32** Suppose $I = \Delta^1$ and take $F := \mathrm{Map}_{\Delta^1}$ to be mapping space bifunctor. Let $f : X \to Y$ and $g : X' \to Y'$ be two arrows of $\mathcal{C}$ and $M$ a $\mathcal{C}$-bimodule, then, we have an equivalence

$$\mathrm{Nat}(\mathrm{Map}_{\Delta^1}, (f^{\mathrm{op}} \times g)^* M) \simeq M(X, X') \times_{M(X,Y')} M(Y, Y')$$

where the maps of the pullback are induced by $f$ and $g$ under $M$.

More generally, if $I = [n]$ and $F = \mathrm{Map}_{[n]}$ its mapping space, then $\mathrm{Map}_{[n]} : [n]^{\mathrm{op}} \times [n] \to \mathcal{S}$ is classified by the right fibration[9] $\mathrm{TwAr}([n]) \to [n] \times [n]^{\mathrm{op}}$ hence we have

$$\mathrm{Nat}(\mathrm{Map}_{[n]}, (f^{\mathrm{op}} \times g)^* M) \simeq \lim_{(i \leq j) \in \mathrm{TwAr}([n])^{\mathrm{op}}} M(f(i), g(j))$$

which recovers the above when $n = 1$. ■

If $\mathcal{C}$ is a stable category, then the tensor with some category $I$ is given by the the smallest stable subcategory $\mathcal{C}_I$ of $\mathrm{Fun}(I^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C}))$ containing the left Kan extension $L_{i,X}$ of functors : $\{i\} \to \mathrm{Ind}(\mathcal{C})$ along $\{i\} \subset I^{\mathrm{op}}$ (see [Sau23a, Proposition 2.2] where $\mathcal{C}_I$ is denoted $I \otimes \mathcal{C}$, and the upcoming [LNS25] where such ideas are greatly expanded upon via the formalism of oplax colimits). There is a canonical functor $\phi : \mathcal{C} \times I \to \mathcal{C}_I$ sending $(X, i)$ to $L_{i,X}$.

**Proposition 3.33** For every $(I, F) \in \mathbf{Cat}^b$ and $(\mathcal{C}, M) \in \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, the object $(\mathcal{C}, M)_{(I,F)}$ exists and its underlying stable category is $\mathcal{C}_I$. Moreover, if $p : \mathcal{C} \times I \to \mathcal{C}$ is the projection, then the associated bimodule is given by the left Kan extension of $M \circ (p^{\mathrm{op}} \times p)$ along $\phi^{\mathrm{op}} \times \phi$. Explicitly, for $f, g \in \mathcal{C}_I$, this is the following colimit:

$$M_{(I,F)}(f, g) \simeq \operatorname*{colim}_{i \in I} M(f(i), g(i))$$

where we wrote $M$ for its extension to a colimit-preserving $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes^L \mathrm{Ind}(\mathcal{C}) \to \mathrm{Sp}$.

**Proof.** Again, we can adopt the same proof strategy as in Proposition 3.26: we have identified the correct underlying stable categories and we are reduced to checking an equivalence of spaces of natural transformations. Using that left Kan extension is left adjoint to precomposition reduces the claim to the same equivalence, *mutatis mutandis*. □

# 4 The K-theory of laced categories

## 4.1 Additivity and semi-orthogonal decompositions

Recall that a semi-orthogonal decomposition of a stable category $\mathcal{C}$ is the datum of a pair subcategories $(\mathcal{A}, \mathcal{B})$ satisfying the following two conditions:

[9]This definition of the twisted arrow category follows the convention of [Lur08, Definition 5.2.1].

(Decomposition) Every $X \in \mathcal{C}$ fits in an exact sequence $A \to X \to B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(Semi-orthogonality) For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\mathrm{map}_{\mathcal{C}}(A, B) \simeq 0$

The semi-orthogonality condition ensures the decomposition of the first condition is unique, and thus functorial, so that what we call a semi-orthogonal decomposition is also what Lurie calls a recollement in [Lur17a, Appendix A.8]. In particular, such decompositions are equivalent to the datum of a semi-split Verdier sequence, i.e. a null-composite sequence

$$\mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{B}$$

such that $i$ is fully-faithful and has a right adjoint and $p$ has a fully-faithful right adjoint. Passing to right adjoints also gives a semi-split Verdier sequence which is split in the other direction, so this presentation has the advantage of not choosing a side in what is usually called left-split and right-split Verdier sequences.

We give the following definition in the laced setting:

**Definition 4.1** Let $(\mathcal{C}, M)$ be a laced category. A laced semi-orthogonal decomposition of $(\mathcal{C}, M)$ is a pair of laced categories $((\mathcal{A}, N); (\mathcal{B}, P))$ with the following extra data and conditions:

(Underlying) The underlying pair of stable categories $(\mathcal{A}, \mathcal{B})$ is a semi-orthogonal decomposition of $\mathcal{C}$.

(Laced sub-categories) The laced functors $(i, \alpha) : (\mathcal{A}, N) \to (\mathcal{C}, M)$ and $(j, \beta) : (\mathcal{B}, P) \to (\mathcal{C}, M)$ induce equivalences $\alpha : N \simeq M \circ (i^{\mathrm{op}} \times i)$ and $\beta : P \simeq M \circ (j^{\mathrm{op}} \times j)$.

(Laced semi-orthogonality) For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $M(A, B) \simeq 0$.

**Lemma 4.2** Let $((\mathcal{A}, N); (\mathcal{B}, P))$ be a laced semi-orthogonal decomposition $(\mathcal{C}, M)$. Then, the left adjoint $p$ of the inclusion $j : \mathcal{B} \to \mathcal{C}$ upgrades to a laced functor $(p, \eta) : (\mathcal{C}, M) \to (\mathcal{B}, P)$ such that

$$(\mathcal{A}, N) \xrightarrow{(i,\alpha)} (\mathcal{C}, M) \xrightarrow{(p,\eta)} (\mathcal{B}, N)$$

is a fiber-cofiber sequence in $\mathbf{TCat}^{\mathrm{Ex}}$.

**Proof.** It holds that the sequence

$$\mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{B}$$

is a fiber-cofiber sequence. To check its enhancement to $\mathbf{TCat}^{\mathrm{Ex}}$ is a fiber-cofiber, it suffices to check that $\alpha : N \to M \circ (i^{\mathrm{op}} \times i)$ is an equivalence and that the mate of $\eta$, $\widehat{\eta} : (p^{\mathrm{op}} \times p)_! M \to N$ is also an equivalence. The former is clear and the second follows from the construction of $\eta$ that we are yet to give.

Remark that since $p$ is left adjoint to $j$, left Kan extension along $p^{\mathrm{op}} \times p$ is computed by precomposing by $j^{\mathrm{op}} \times j$ so we can take $\widehat{\eta}$ to be $\beta^{-1} : M \circ (j^{\mathrm{op}} \times j) \to N$, which is indeed an equivalence. □

The next three lemmata provide examples ordered by complexity of laced semi-orthogonal decompositions. First, let us remark that our definition encompasses the usual semi-orthogonal decompositions:

**Lemma 4.3** Let $(\mathcal{A}, \mathcal{B})$ be a semi-orthogonal decomposition of $\mathcal{C}$. Then, $((\mathcal{A}, \mathrm{Map}_{\mathcal{A}}); (\mathcal{B}, \mathrm{Map}_{\mathcal{B}}))$ is a laced semi-orthogonal decomposition of $(\mathcal{C}, \mathrm{Map}_{\mathcal{C}})$.

**Proof.** Since $\mathcal{A}$ and $\mathcal{B}$ are full subcategories, the mapping space functor are indeed restricted along the inclusions, whereas the laced semi-orthogonality condition is simply semi-orthogonality of the underlying stable decomposition. □

Our first non-trivial example of the laced situation concerns the cotensor $(\mathcal{C}, M)^{([1], \mathrm{Map})}$ from Example 3.32. We claim that $(\mathcal{C}, M)^{([1], \mathrm{Map})}$ plays the role of the arrow category $\mathcal{C}^{[1]}$ in the laced world, and can be obtained as the semi-orthogonal decomposition of two copies of $(\mathcal{C}, M)$. In fact, the symmetry implies that the decomposition is fully orthogonal, and adapting the proof of the hermitian case (see Theorem 1.2.9 of [CDH+23b]), one could show that every fully-orthogonal decomposition can be obtained by pulling back the target projection.

**Lemma 4.4** Let $(\mathcal{C}, M)$ be a laced category. Then, $((\mathcal{C}, M); (\mathcal{C}, M))$ is a semi-orthogonal decomposition of $(\mathcal{C}, M)^{([1],\mathrm{Map})}$.

**Proof.** It is a classical fact that $\mathcal{C}^{[1]}$ is the semi-orthogonal decomposition of two copies of $\mathcal{C}$, the inclusions being given by $i : X \mapsto (X \to 0)$ and $j : X \mapsto \mathrm{id}_X$. Recall from Example 3.32 that the bimodule of $(\mathcal{C}, M)^{([1],\mathrm{Map})}$ fits inside the following cartesian square:

$$\begin{tikzcd} M^{([1],\mathrm{Map})}(f : X \to Y, g : X' \to Y') \arrow[r] \arrow[d] & M(Y, Y') \arrow[d, "M(f,-)"] \\ M(X, X') \arrow[r, "M(-,g)"] & M(X, Y') \end{tikzcd}$$

Now, laced decomposition follows from plugging $Y = Y' = 0$ which makes the vertical maps into equivalences, and plugging $f, g$ being identities which makes every map into equivalences, so that in both cases $M^{([1],\mathrm{Map})}$ recovers $M$. If one plugs $Y = 0$ and $g$ the identity of say $X'$, then horizontal maps are equivalences so in particular, $M^{([1],\mathrm{Map})}$ vanishes which yields laced semi-orthogonality. □

Let us conclude this series of examples by a case which is not symmetric, in fact even not fully-orthogonal, and concerns the cotensor $(\mathcal{C}, M)^{([1],*)}$ which was considered in example 3.31.

**Lemma 4.5** Let $(\mathcal{C}, M)$ be a laced category. Then, $((\mathcal{C}, 0); (\mathcal{C}, M))$ is a semi-orthogonal decomposition of $(\mathcal{C}, M)^{([1],*)}$.

**Proof.** The underlying part is the same as the previous lemma, and this time, we have the following formula:

$$M^{([1],*)}(f : X \to Y, g : X' \to Y') \simeq M(Y, X')$$

It follows that if $Y = Y' = 0$, $M^{([1],*)}$ vanishes whereas it recovers $M$ in the case where $f, g$ are identities, so that we have laced decomposition. In fact, $M^{([1],*)}$ vanishes as soon as $Y = 0$ so that we have also laced semi-orthogonality. □

**Proposition 4.6** The two adjoint functors $L : \mathcal{C} \mapsto (\mathcal{C}, \mathrm{Map})$ and $\mathrm{Lace} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ preserve semi-orthogonal decompositions.

**Proof.** For $L$, this is the content of Lemma 4.3. Let $((\mathcal{A}, N); (\mathcal{B}, P))$ be a laced semi-orthogonal decomposition of $(\mathcal{C}, M)$; here we will view $M$ as an exact functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ thanks to Remark 3.6. We want to show that $\mathrm{Lace}(\mathcal{C}, M)$ admits $(\mathrm{Lace}(\mathcal{A}, N), \mathrm{Lace}(\mathcal{B}, P))$ as a semi-orthogonal decomposition.

Let $X \in \mathcal{C}$ and $f : X \to M(X)$ so that $(X, f)$ is any object of $\mathrm{Lace}(\mathcal{C}, M)$. Since $(\mathcal{A}, \mathcal{B})$ is an orthogonal decomposition of $\mathcal{C}$, we have an exact sequence $q(X) \to X \to p(X)$ where $p : \mathcal{C} \to \mathcal{B}$ and $q : \mathcal{C} \to \mathcal{A}$ are the localisation functors and we omit to write the inclusions. Remark that in this situation, the laced sub-category axiom implies that the space of maps $q(X) \to N(q(X))$ and $q(X) \to M(q(X))$ are equivalent (and dually for $P$ and $p$). In particular, it follows that the following diagram commutes and its bottom horizontal row is exact:

$$\begin{tikzcd} q(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & p(X) \arrow[d] \\ N(q(X)) \arrow[r] & M(X) \arrow[r] & P(p(X)) \end{tikzcd}$$

This is exactly the datum of an exact sequence in $\mathrm{Lace}(\mathcal{C}, M)$ whose first term is in $\mathrm{Lace}(\mathcal{A}, N)$, last term is in $\mathrm{Lace}(\mathcal{B}, P)$ and middle term is $(X, f)$, hence we have decomposition.

For the semi-orthogonality, recall that mapping spaces in $\mathrm{Lace}(\mathcal{C}, M)$ can be expressed as equalizers by combining 3.7 and [NS17], so that we are reduced to show that for every $(A, f) \in \mathrm{Lace}(\mathcal{A}, N)$, $(B, g) \in \mathrm{Lace}(\mathcal{B}, P)$, the following spectrum vanishes:

$$\mathrm{Eq}\left( \mathrm{map}_{\mathcal{C}}(A, B) \underset{M(g)\circ(-)}{\overset{M(-)\circ f}{\rightrightarrows}} \mathrm{map}_{\mathrm{Ind}\,\mathcal{C}}(A, M(B)) \right)$$

This follows from the fact that both $\mathrm{map}_{\mathcal{C}}(A, B)$ and $\mathrm{map}(A, M(B))$ vanish, the former by semi-orthogonality of the underlying $(\mathcal{A}, \mathcal{B})$ and the latter by laced semi-orthogonality. □

Finally, let us conclude by a few words on laced (semi-)additive invariants.

**Definition 4.7** Let $\mathcal{E}$ be a stable category. We say that a functor $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is *laced additive* or *laced split-Verdier localizing* if it sends semi-orthogonal decompositions to direct sum decompositions in $\mathcal{E}$.

## 4.2 The universal property of laced K-theory

From the usual K-theory functor $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$, the most natural way of producing a functor from the tangent bundle is to left Kan extend along the cotangent complex $L$. Since $L$ is left adjoint to Lace, it follows that this left Kan extension is computed by precomposition by Lace, hence the following definition.

**Definition 4.8** We let $\mathrm{K}^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ denote the composite

$$\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \xrightarrow{\mathrm{Lace}} \mathbf{Cat}^{\mathrm{Ex}} \xrightarrow{K} \mathrm{Sp}$$

It comes naturally equipped with a natural transformation $\Sigma^{\infty}_{+} \mathrm{Lace}^{\simeq} \to \mathrm{K}^{\mathrm{lace}}$.

By Proposition 4.6, $\mathrm{K}^{\mathrm{lace}}$ is laced-additive, since $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is and Lace preserves orthogonal decompositions. In fact, the K-theory functor enjoys a universal property with respect to additive functors, which we can obtain by reformulating the main result of [BGT13]:

**Proposition 4.9 — Blumberg-Gepner-Tabuada.** The natural transformation $\Sigma^{\infty}_{+} \iota \to \mathrm{K}$ of functors $\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ exhibits algebraic K-theory as the initial additive invariant under $\Sigma^{\infty}_{+} \iota$.

**Proof.** Recall from [BGT13] that there is a stable category $\mathcal{M}_{\mathrm{add}}$ of additive motives given with a functor $U_{\mathrm{add}} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{M}_{\mathrm{add}}$ such that for every stable $\mathcal{E}$, there is an equivalence

$$U^{*}_{\mathrm{add}} : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{M}_{\mathrm{add}}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{add}}(\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$$

where the right hand side denotes additive functors $\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$. Remark that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{M}_{\mathrm{add}}, \mathcal{E}) \arrow[r, "U^{*}_{\mathrm{add}}"] & \mathrm{Fun}(\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \\
\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{M}_{\mathrm{add}}, \mathcal{E}) \arrow[u, "\subset"] \arrow[r, "\simeq"] & \mathrm{Fun}_{\mathrm{add}}(\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \arrow[u, "\subset"]
\end{tikzcd}$$

and every arrow has a left adjoint. By unicity, the diagram of left adjoints also commutes and it follows that if $F$ is any functor $\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$, then the initial additive invariant under $F$ is given by $(\mathrm{P}_1((U_{\mathrm{add}})_! F)) \circ U_{\mathrm{add}}$, where $(-)_!$ indicates left Kan extension along $(-)$ and $\mathrm{P}_1 : \mathrm{Fun}(\mathcal{M}_{\mathrm{add}}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{M}_{\mathrm{add}}, \mathcal{E})$ is the first Goodwillie derivative.

Left Kan extension along $f$ sends $\mathrm{Map}(X, -)$ to $\mathrm{Map}(f(X), -)$ and is a colimit-preserving operation. Applied to $\Sigma^{\infty}_{+} \iota$, which is the infinite-suspension of $\mathrm{Map}(\mathrm{Sp}^{\mathrm{fin}}, -)$, the left Kan extension along $U_{\mathrm{add}}$ yields $\Sigma^{\infty}_{+} \mathrm{Map}(U_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}}), -)$. Taking the first Goodwillie derivative yields the spectrally-corepresented functor $\mathrm{map}(U_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}}), -)$; the argument is eventually spelled out in Lemma 5.23 and uses that $\mathrm{map}(X, -)$ is exact and satisfies $\Omega^{\infty} \mathrm{map}(X, -) \simeq \mathrm{Map}$. But the following has been shown in [BGT13]:

$$\mathrm{K}(\mathcal{C}) \simeq \mathrm{map}_{\mathcal{M}_{\mathrm{add}}}(U_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}}), U_{\mathrm{add}}(\mathcal{C}))$$

hence K is the wanted functor. This concludes. □

Actually $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ has a stronger property: it is Verdier localizing and in particular, semi-split Verdier localizing, so it is also universal as a semi-additive invariant. It follows from purely formal manipulations that $\mathrm{K}^{\mathrm{lace}}$ also enjoys a universal property with respect to (laced semi-)additive functors.

**Theorem 4.10** The natural transformation $\Sigma^\infty_+ \mathrm{Lace}^\simeq \to \mathrm{K}^{\mathrm{lace}}$ of functors $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ exhibits laced K-theory as the initial additive invariant under $\Sigma^\infty_+ \mathrm{Lace}^\simeq$.

**Proof.** This is a completely formal corollary from the above proposition and the following fact: the adjunction between $L$ and Lace preserves orthogonal decomposition by Proposition 4.6. Indeed, firstly, since Lace preserves additive sequences and $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is additive, so is $\mathrm{K}^{\mathrm{lace}}$. Using the adjunction between the precomposition functors $L^*$ and $\mathrm{Lace}^*$, for every functor $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$, there is an equivalence

$$\mathrm{Nat}(\Sigma^\infty_+ \iota \,\mathrm{Lace}, F) \simeq \mathrm{Nat}(\Sigma^\infty_+ \iota, F \circ L)$$

If $F$ is additive, so is $F \circ L$ hence by Theorem 4.9, the right hand side is equivalent to $\mathrm{Nat}(K, F \circ L)$ which is itself equivalent to $\mathrm{Nat}(\mathrm{K}^{\mathrm{lace}}, F)$ using the adjunction between $L^*$ and $\mathrm{Lace}^*$ again. This concludes. □

**Remark 4.11** In the laced setting, the construction of non-commutative additive motives of [BGT13] or the Poincaré version of [CDH+23d] can be adapted to give a presentable stable $\mathcal{M}^{lace}_{\mathrm{add}}$ and a universal laced-additive $U^{lace}_{\mathrm{add}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Mot}^{lace}_{\mathrm{add}}$. As in *loc. cit.*, this is done using Proposition 5.3.6.2 of [Lur08], and then taking the Spannier-Whitehead stabilization, once set-theoretic problems have been dealt with. Alternatively, one could show that those motives are the stable envelope of an additive category obtained as the localization of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. Since this category of motives will not play a role in the subsequent sections, we will save ourselves the trouble and delay this construction to later work.

Supposing $\mathrm{Mot}^{lace}_{\mathrm{add}}$ built, the proof of Theorem 4.10 can be worked backwards to give a similar formula for $\mathrm{K}^{\mathrm{lace}}$ as Blumberg-Gepner-Tabuada's formula for K-theory.

With monoidality, one can deal away with the natural transformation. Recall the following result [BGT14, Theorem 1.5], for which we give a model-independent proof:

**Proposition 4.12** The K-theory functor K admits a lax-monoidal refinement which makes it the initial lax-monoidal additive functor $\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$.

**Proof.** The category of additive motives $\mathrm{Mot}_{\mathrm{add}}$ admits a stable symmetric monoidal structure such that $U_{\mathrm{add}} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Mot}_{\mathrm{add}}$ is symmetric monoidal and thus the unit is given by $U_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}})$; this follows from the universal property if we can check that the composite

$$\mathbf{Cat}^{\mathrm{Ex}} \times \mathbf{Cat}^{\mathrm{Ex}} \xrightarrow{\otimes} \mathbf{Cat}^{\mathrm{Ex}} \xrightarrow{U_{\mathrm{add}}} \mathrm{Mot}_{\mathrm{add}}$$

is additive in each variable; but this is clear because $- \otimes \mathcal{C}$ preserves adjunctions and cofiber sequences hence split-Verdier sequences. In consequence, Corollary 6.8 of [Nik16] implies that $\mathrm{map}(U_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}}), -)$ is the initial object of $\mathrm{Fun}^{\mathrm{Ex}}_{\mathrm{lax}}(\mathcal{C}, \mathrm{Sp})$. Since $U_{\mathrm{add}}$ is a symmetric monoidal, it induces an equivalence

$$\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Mot}_{\mathrm{add}}, \mathrm{Sp})^{\otimes} \xrightarrow{U^*_{\mathrm{add}}} \mathrm{Fun}_{\mathrm{add}}(\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{Sp})^{\otimes}$$

of the symmetric monoidal refinements of the functor categories. In particular, this equivalence descends to the category of algebra objects, i.e. lax-monoidal functors, which implies the result. □

By Corollary 3.21, the cotangent complex functor $L : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is a (strong) monoidal functor so that its right adjoint Lace inherits a lax-monoidal structure, which induces a lax-monoidal refinement of $\mathrm{K}^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$.

**Proposition 4.13** The functor $\mathrm{K}^{\mathrm{lace}}$ is the initial lax-monoidal additive functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$.

**Proof.** By the above discussion, the adjunction between $L$ and Lace induces a functor

$$\mathrm{Lace}^* : \mathrm{Alg}(\mathrm{Fun}_{\mathrm{add}}(\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{Sp})^{\otimes}) \longrightarrow \mathrm{Alg}(\mathrm{Fun}_{\mathrm{add}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{Sp})^{\otimes})$$

which is left adjoint to $L^*$. In particular, it preserves the initial object which is K-theory so that the claim follows. $\square$

## 4.3 Flavours of laced-Verdier sequences

We dedicate this section to talking about the equivalent of Verdier and Karoubi sequences in the laced setting. The characterizations of Proposition 3.16 assemble to give a characterization of null-composite sequences which are both fiber and cofiber sequences in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$;

**Definition 4.14** A sequence $e : (\mathcal{C}, F) \xrightarrow{(i,\alpha)} (\mathcal{D}, G) \xrightarrow{(p,\beta)} (\mathcal{E}, H)$ is a *naive laced-Verdier sequence* if it is a fiber and a cofiber sequence in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. Equivalently, this is the following conditions:

- The sequence of stable categories $\mathcal{C} \longrightarrow \mathcal{D} \longrightarrow \mathcal{E}$ is a Verdier sequence.
- The natural transformation $\alpha : F \xrightarrow{\simeq} G \circ (i^{\mathrm{op}} \times i)$ and $\hat{\beta} : (p^{\mathrm{op}} \times p)_! G \xrightarrow{\simeq} H$ are equivalences, where $\hat{\beta}$ is the mate of $\beta$.

For abstract reasons, there exists a universal laced-Verdier localizing invariant under $\Sigma^\infty_+ \mathrm{Lace}^{\simeq}$. However, unlike in the split case, it need not be that Lace preserves Verdier sequences and thus, there is no reason to expect that $\mathrm{K}^{\mathrm{lace}} = K \circ \mathrm{Lace}$ would be the invariant obtained by the previous abstract procedure.

We can try to circumvent this issue by specifically adding to the conditions of Definition 4.14 that Lace sends such sequences to Verdier sequences. Though this fixes the problem in K-theory, this is unfortunately too *ad-hoc* in general and we will need an even stronger condition so that the definition plays nicely with the functor $\mathrm{THH} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ which will be built in the next section.

**Definition 4.15** A sequence $e : (\mathcal{C}, F) \xrightarrow{(i,\alpha)} (\mathcal{D}, G) \xrightarrow{(p,\beta)} (\mathcal{E}, H)$ is a *fine laced-Verdier sequence* if it is a naive Verdier sequence such that the following condition is realized:

- For every $n \geq 0$, the sequence

$$\mathrm{Lace}(\mathcal{C}, \Sigma^n F) \longrightarrow \mathrm{Lace}(\mathcal{D}, \Sigma^n G) \longrightarrow \mathrm{Lace}(\mathcal{E}, \Sigma^n H)$$

is a Verdier sequence.

A functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ which sends fine laced-Verdier sequences to exact sequences in a stable $\mathcal{E}$ will be called *weakly laced-Verdier localizing.*

In particular, fine laced-Verdier sequence are laced-Verdier sequences. Hence, as the name suggests, laced-Verdier localizing functors are in particular weakly laced-Verdier localizing.

**Proposition 4.16** The functor $\mathrm{K}^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is weakly laced-Verdier localizing. Consequently, it is also the initial such functor under $\Sigma^\infty_+ \mathrm{Lace}^{\simeq}$.

**Proof.** Thanks to Proposition 2.4, the functor $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is Verdier-localizing (the proof of this fact can be found for instance in [Sau23a, Theorem 3.13] or [HLS23, Theorem 6.1] for a minimal argument), hence this is immediate. $\square$

Thanks to Corollary 5.34, having put the suspensions in the previous definition will imply that $\mathrm{THH} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is weakly laced-Verdier localizing as such functors are of course stable under filtered colimits.

# 5 Trace-like functors and THH of laced categories

In the previous sections, we have built a category $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ and extended algebraic K-theory into a functor $\mathrm{K}^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ which we have characterized by a universal property. This could have been achieved in $\mathbf{Cat}^{\mathrm{Ex}}$ already; in fact the previous section would be shallow without the universal property of K-theory as a functor from stable categories. However, the salient feature of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is that it will allow to treat THH on equal footing with laced K-theory.

## 5.1 A primer on THH of stable categories

As in the case of K-theory, to make the extension of THH to laced categories, it is easiest to first reformulate the definitions in the stable case. The purpose of this section is thus to set-up different definitions of classical objects, which will extend in a straightforward manner to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. The results here are folklore for which we could not find a reference.

Before we delve into THH proper, let us introduce an intermediate object called unstable topological Hochschild homology, defined for any category.

**Definition 5.1** Let $\mathcal{C}$ be any category. The *unstable topological Hochschild homology* of $\mathcal{C}$ is the space $\mathrm{uTHH}(\mathcal{C})$ given by the following coend:

$$\mathrm{uTHH}(\mathcal{C}) = \int^{X\in\mathcal{C}} \mathrm{Map}_{\mathcal{C}}(X,X)$$

where $\mathrm{Map}_{\mathcal{C}}$ denote the mapping space of $\mathcal{C}$. This defines a functor $\mathrm{uTHH} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{S}$.

The intuition behind this formula is the following: the bivariant Yoneda lemma tells us for a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$, $\mathrm{Nat}(\mathrm{Map}, F)$ is equivalent to the end $\int_{X\in\mathcal{C}} F(X,X)$. The dual construction hence corresponds to $\mathrm{Map} \otimes_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} F$. When $F$ is taken to be Map itself, this mimics the formula $R \otimes_{R^{\mathrm{op}}\otimes R} R$ which defines $\mathrm{THH}(R)$ for a ring spectrum $R$ in an unstable manner.

**Remark 5.2** Let $\mathcal{C}$ be a category. We have an equivalence

$$\mathrm{uTHH}(\mathcal{C}) \simeq \operatorname*{colim}_{f:X\to Y\in\mathrm{TwAr}(\mathcal{C})} \mathrm{Map}_{\mathcal{C}}(Y,X)$$

This is by fiat, following [GHN17, Definition 2.6]; recall that we defined the twisted arrow category so that $\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}\times\mathcal{C}^{\mathrm{op}}$ is the right fibration classifying $\mathrm{Map}_{\mathcal{C}}$ so that the convention of *loc. cit.* coincides with our own.

In particular, if $\mathcal{C}$ is a space, then we have $\mathrm{Ar}(\mathcal{C}) \simeq \mathrm{TwAr}(\mathcal{C})$ and every arrow in the previous colimit can be taken to be a self-equivalence $X \to X$. Thus, for a space $X$, $\mathrm{uTHH}(X)$ is equivalently given by the mapping space $\mathrm{Map}(S^1, X)$, i.e. the free loop space on $X$. In particular, the following composite is natural:

$$\iota\mathcal{C} \to \mathrm{Fun}(S^1, \iota\mathcal{C}) \simeq \mathrm{uTHH}(\iota\mathcal{C}) \to \mathrm{uTHH}(\mathcal{C})$$

and thus provides a natural transformation $\iota \to \mathrm{uTHH}$.

The stabilized version of unstable topological Hochschild homology is none other than the usual THH. In fact, we will prove in the more general setting of laced category (see 5.28) that the stabilization of uTHH, in the sense of Goodwillie calculus, is indeed given by THH.

**Definition 5.3** Let $\mathcal{C}$ be a stable category; in particular $\mathcal{C}$ is enriched in spectra and we denote map the enriched mapping object. The *topological Hochschild homology* of $\mathcal{C}$ is the spectrum $\mathrm{THH}(\mathcal{C})$ given by the following coend:

$$\mathrm{THH}(\mathcal{C}) = \int^{X\in\mathcal{C}} \mathrm{map}_{\mathcal{C}}(X,X)$$

This defines a functor $\mathrm{THH} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ to spectra, equipped with a canonical natural transformation $\mathrm{uTHH} \to \Omega^\infty \mathrm{THH}$, or equivalently $\Sigma^\infty_+ \mathrm{uTHH} \to \mathrm{THH}$.

As for its unstable counterpart, one has to understand this coend as perfoming some sort of relative tensor product map $\otimes_{\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}}$ map, though this time, we will show it actually recovers the classical $\mathrm{THH}(R) := R \otimes_{R^{\mathrm{op}}\otimes R} R$.

**Remark 5.4** As in the previous remark, we have the following formula for $\mathrm{THH}(\mathcal{C})$:

$$\mathrm{THH}(\mathcal{C}) \simeq \operatorname*{colim}_{f:X\to Y\in\mathrm{TwAr}(\mathcal{C})} \mathrm{map}_{\mathcal{C}}(Y,X)$$

and the map $\Sigma^\infty_+ \mathrm{uTHH} \to \mathrm{THH}$ is induced by taking the colimit over $\mathrm{TwAr}(\mathcal{C})$ of the natural transformation $\Sigma^\infty_+ \mathrm{Map} \to \mathrm{map}$.

**Remark 5.5** This definition of THH is not the usual one in the literature for stable categories; for instance, the formula in Section 10 of [BGT13] is both more intricate and closer to the cyclic bar construction of THH for rings, as in Definition III.2.3 of [NS17] or the more recent version with coefficients in [KMN23].

We will show in the next section that the different formula agree. Namely, let $\mathcal{C}$ be a category. Then, Proposition 5.10 will show that there is an equivalence

$$\mathrm{uTHH}(\mathcal{C}) \simeq \left| \cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \operatorname*{colim}_{X,Y\in\iota\mathcal{C}} \mathrm{Map}_{\mathcal{C}}(X,Y)\times\mathrm{Map}_{\mathcal{C}}(Y,X) \rightrightarrows \operatorname*{colim}_{X\in\iota\mathcal{C}} \mathrm{Map}_{\mathcal{C}}(X,X) \right|$$

with face maps given by suitable compositions. When $\mathcal{C}$ is further stable, we will explain in Remark 5.29, using Proposition 5.28, that there is a second equivalence:

$$\mathrm{THH}(\mathcal{C}) \simeq \left| \cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \operatorname*{colim}_{X,Y\in\mathcal{C}} \mathrm{map}_{\mathcal{C}}(X,Y)\otimes\mathrm{map}_{\mathcal{C}}(Y,X) \rightrightarrows \operatorname*{colim}_{X\in\mathcal{C}} \mathrm{map}_{\mathcal{C}}(X,X) \right|$$

In particular, our definition of THH is the same as that of [BGT13] or [HSS17].

**Remark 5.6** Let $\mathcal{C}$ be stable and consider the composition:

$$\iota\mathcal{C} \to \mathrm{Fun}(S^1,\iota\mathcal{C}) \simeq \mathrm{uTHH}(\iota\mathcal{C}) \to \mathrm{uTHH}(\mathcal{C}) \to \Omega^\infty\,\mathrm{THH}(\mathcal{C})$$

This map is natural in $\mathcal{C}$, thus induces a natural transformation $\Sigma^\infty_+\iota \to \mathrm{THH}$. The Bökstedt trace $\mathrm{K}\to\mathrm{THH}$ is then obtained by the universal property of K, using that THH is additive (see [BGT13, Proposition 10.2]).

## 5.2 Laced THH, trace-equivalences and the cyclic bar construction

The advantage of the definition of THH we gave in the previous section is that it already features a bimodule, namely $\mathrm{map}_{\mathcal{C}}$. By simply replacing it with any bimodule $M : \mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}\to\mathrm{Sp}$, we get a definition of THH adapted to laced categories:

**Definition 5.7** Let $(\mathcal{C}, M)$ be a laced category. The *topological Hochschild homology* of $(\mathcal{C}, M)$ is the spectrum $\mathrm{THH}(\mathcal{C}, M)$ given by the following coend:

$$\mathrm{THH}(\mathcal{C}, M) = \int^{X\in\mathcal{C}} M(X,X) \in \mathrm{Sp}$$

This defines a functor $\mathrm{THH} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$. More generally, for pairs $(I, F)$ with $I$ a category and $F$ a functor $F : I^{\mathrm{op}}\times I\to\mathcal{S}$, we can also define the *unstable topological Hochschild homology* by:

$$\mathrm{uTHH}(I, F) = \int^{X\in I} F(X,X) \in \mathcal{S}$$

The functor $\mathrm{uTHH} : \mathbf{Cat}^b \to \mathcal{S}$ is such that, when restricted to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, we have a natural transformation $\Sigma^\infty_+\,\mathrm{uTHH}\to\mathrm{THH}$.

Comparing the above definition with Definition 5.3, we see that $\mathrm{THH}(\mathcal{C}) \simeq \mathrm{THH}(\mathcal{C}, \mathrm{map})$. In particular, the natural transformation $\beta : \mathrm{K} \to \mathrm{THH}$ of functors $\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ yields a natural transformation $\mathrm{K}^{\mathrm{lace}} \to \mathrm{THH}$, which we are tempted to call the *laced Bökstedt trace*, given by:

$$\mathrm{K}(\mathrm{Lace}(\mathcal{C}, M)) \xrightarrow{\beta_{\mathrm{Lace}(\mathcal{C},M)}} \mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), \mathrm{map}) \longrightarrow \mathrm{THH}(\mathcal{C}, M)$$

where the second map is induced by the counit of the adjunction $L \dashv \mathrm{Lace}$. Let us stress that the second map is not an equivalence, as we have $\mathrm{THH}(\mathcal{C}, 0) \simeq 0$ but $\mathrm{THH}(\mathrm{Lace}(\mathcal{C}, 0), \mathrm{map}) \simeq \mathrm{THH}(\mathcal{C})$. This map will be thoroughly investigated in §8.

**Lemma 5.8** We have an equivalence of spectra

$$\mathrm{THH}(\mathcal{C}, M) \simeq \operatorname*{colim}_{f:X\to Y\in\mathrm{TwAr}(\mathcal{C})} M(Y, X)$$

which is natural in $(\mathcal{C}, M)$. Similarly, for spaces and uTHH restricted to $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$:

$$\mathrm{uTHH}(\mathcal{C}, M) \simeq \operatorname*{colim}_{f:X\to Y\in\mathrm{TwAr}(\mathcal{C})} \Omega^\infty M(Y, X)$$

**Proof.** As in Remarks 5.2 and 5.6, this is by fiat following [GHN17, Definition 2.6]. ☐

The above definition is unusual, in the sense that THH is usually defined as the result of a cyclic bar construction, i.e. the geometric realization of a cyclic object which looks like the Bar construction where some terms have been modified to be *cyclic*. We will recover such an expression by combining the previous lemma and the Bousfield-Kan formula. Moreover, in the laced realm, this cyclic bar construction can be characterized as the universal way to enforce a property on a functor $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$. We call this property being *trace-like*, for reasons that will only come to light in the following sections.

Let us first build the layers of our cyclic bar construction. If $(\mathcal{C}, M)$ is a laced category, then it follows from Example 3.31 that the objects of the category $\mathrm{Lace}((\mathcal{C}, M)^{([n],*)})$ are chains $X_0 \to \ldots \to X_n$ in $\mathcal{C}$ and a point in $\Omega^\infty M(X_n, X_0)$, which we can equivalently see as an arrow $X_n \to M(X_0)$ in $\mathrm{Ind}\,\mathcal{C}$.

Hence, $\mathrm{Lace}((\mathcal{C}, M)^{([n],*)})$ looks like a category of $n$-cycles where $n-1$ terms are simply arrows $X_i \to X_{i+1}$, i.e points in $\mathrm{Map}(X_i, X_{i+1})$, and the last term is "cycling back", but in a twisted way, giving a point in $\mathrm{Map}(X_n, M(X_0))$. When $M = \mathrm{map}_{\mathcal{C}}$, this is exactly the data of the $n$-th simplicies of the cyclic bar construction of Remark 5.5. Hence, we feel justified to give the following definition:

**Definition 5.9** Let $(\mathcal{C}, M)$ be a laced category, we denote $\mathrm{Bar}_n(\mathcal{C}, M)$ the following space :

$$\mathrm{Bar}_n(\mathcal{C}, M) := \mathrm{Lace}^{\simeq}((\mathcal{C}, M)^{([n],*)})$$

By functoriality of the cotensor, we have a simplicial space $\mathrm{Bar}_\bullet(\mathcal{C}, M) : \Delta^{\mathrm{op}} \to \mathcal{S}$ and in fact, a functor $\mathrm{Bar}_\bullet : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{S}^{\Delta^{\mathrm{op}}}$. We call it the *cyclic bar construction* of the functor $\mathrm{Lace}^{\simeq}$, although we remark that it need not be a cyclic object in general.

This object has actually a lot more structure which is not recorded by the simplicial structure and it will be the main ordeal of later sections to study systematically this extra structure (or more precisely, this is systematically done in [HNS25] because we will only sketch some of the details of this structure); for now, we will be content with the above definition, and the fact that it recovers unstable THH:

**Proposition 5.10** Let $(\mathcal{C}, M) \in \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. We have a natural equivalence

$$\mathrm{uTHH}(\mathcal{C}, M) \simeq |\,\mathrm{Bar}_\bullet(\mathcal{C}, M)|$$

**Proof.** By Lemma 5.8, $\mathrm{uTHH}(\mathcal{C}, M)$ is given by the colimit of the composite $\Omega^\infty M \circ p$ where $p$ is the right fibration $\mathrm{TwAr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ classifying the mapping space of $\mathcal{C}$. In formula, this is:

$$\mathrm{uTHH}(\mathcal{C}, M) \simeq \operatorname*{colim}_{f:X\to Y\in\mathrm{TwAr}(\mathcal{C})} \Omega^\infty M(Y, X)$$

Let $T_n := \mathrm{Map}(\Delta^n, \mathrm{TwAr}(\mathcal{C}))$, then the inclusion $T_0 \simeq \iota\,\mathrm{TwAr}(\mathcal{C}) \to \mathrm{TwAr}(\mathcal{C})$ induces maps

$$\alpha_n : T_n \xrightarrow{d_0} T_0 \longrightarrow \mathrm{TwAr}(\mathcal{C}) \xrightarrow{\Omega^\infty M \circ p} \mathcal{S}$$

By Corollary 12.5 of [Sha23] (the so-called *Bousfield-Kan formula*), taking the colimit of $\alpha_n$ produces a simplicial object $X_\bullet$ whose geometric realization recovers $\mathrm{uTHH}(\mathcal{C}, M)$.

We now identify each $X_n$ in terms of the cyclic bar construction: the two do not quite coincide, instead we now show that $X_n$ is the $n^{th}$-stage of the edgewise-subdivision of the cyclic bar construction; this is enough to conclude because this subdivision preserves the geometric realization.

The *edgewise-subdivision* is the functor $e : \Delta \to \Delta$ induced by the inclusion $[n] \to [n] * [n]^{\mathrm{op}} \simeq [2n+1]$ in the first factor. For any category $X$, precomposition by $e$ induces an equivalence $\mathrm{TwAr}(X) \simeq X \circ e$ by viewing both as quasi-categories. Moreover, $e$ is a combinatorial subdivision in the sense of [Bar13, 2.2] although we remark that because of different conventions, our edgewise-subdivision does not coincide with that of Barwick. Nonetheless, $e^{\mathrm{op}} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ is cofinal, so that showing that $X_n \simeq \mathrm{Bar}_{e(n)}(\mathcal{C}, M)$ concludes.

By definition, we have $\mathrm{Bar}_{e(n)}(\mathcal{C}, M) \simeq \mathrm{Lace}^{\simeq}((\mathcal{C}, M)^{([n]^{\mathrm{op}}*[n],*)})$. Remark that on the underlying stable categories, we have $\mathcal{C}^{[n]^{\mathrm{op}}*[n]} \simeq \mathrm{TwAr}(\mathcal{C})^{[n]}$; we can also rewrite the bimodule under this equivalence, and we claim the following holds:

$$(\mathcal{C}, M)^{([n]^{\mathrm{op}}*[n],*)} \simeq (\mathrm{TwAr}(\mathcal{C}), M \circ p)^{([n],*)}$$

This is the case when $n = 0$ by Example 3.31. More generally, recall that if $I = \{i_0 < ... < i_\kappa\}$ is linearly ordered, the formula for $M^{(I,*)}$ is the evaluation $M(X(i_\kappa), Y(i_0))$, again by Example 3.31. For $I = [n]^{\mathrm{op}} * [n]$, this is the following diagram where we colored in red the rightmost-vertical arrow because it corresponds to the first point when passing to the twisted-arrow-category point of view:

$$\begin{tikzcd} \bullet & \bullet \arrow[l] & \dots \arrow[l] & \bullet \arrow[l] \arrow[d] & \color{red}\bullet \arrow[l] \arrow[d, red] \\ \bullet \arrow[u, leftarrow] \arrow[r] & \bullet \arrow[u, leftarrow] \arrow[r] & \dots \arrow[r] & \bullet \arrow[r] & \color{red}\bullet \end{tikzcd}$$

In particular, we deduce that $M^{([n]^{\mathrm{op}}*[n],*)} \simeq (M \circ p)^{([n],*)}$. Finally, to recover the colimit defining $X_n$, remark that there is an equivalence

$$\mathrm{Lace}^{\simeq}(\mathcal{D}, N) \simeq \operatorname*{colim}_{X \in \iota\mathcal{D}} \Omega^\infty N(X, X)$$

since $\mathrm{Lace}^{\simeq}(\mathcal{D}, N) \to \iota\mathcal{D}$ is the left fibration classifying $\Omega^\infty N \circ \Delta$, where $\Delta : \iota\mathcal{C} \to \iota\mathcal{C}^{\mathrm{op}} \times \iota\mathcal{C}$ is the diagonal by Lemma 3.13. Applied to the rewriting we produced, we get an equivalence

$$\mathrm{Bar}_{e(n)}(\mathcal{C}, M) \simeq \operatorname*{colim}_{Y \in \iota\,\mathrm{TwAr}(\mathcal{C})^{[n]}} \Omega^\infty M \circ p(Y(0))$$

where we recognize on the right hand side the definition of $X_n$. This concludes. □

Inclusion of the 0-simplices yields a natural transformation $\mathrm{Lace}^{\simeq} \to \mathrm{uTHH}$. Our goal is to show that this natural transformation exhibits its target as the initial functor under $\mathrm{Lace}^{\simeq}$ for a certain property, which we call being trace-like. In fact, we will show more generally that this happens for the natural transformation $F \to \mathrm{cyc}(F)$, where $\mathrm{cyc}(F)$ is cyclic bar construction adapted to a functor $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$, which plays the role of $\mathrm{Lace}^{\simeq}$ in the above story. Let us first explain what are trace-like functors.

**Definition 5.11** Let $f, g : (\mathcal{C}, M) \to (\mathcal{D}, N)$ be two laced functors. A *trace homotopy* from $f$ to $g$ is a functor $H : (\mathcal{C}, M) \to (\mathcal{D}, N)^{([1],*)}$ such that the following diagram commutes:

$$\begin{tikzcd} & & (\mathcal{D}, N) \\ (\mathcal{C}, M) \arrow[r, "H"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & (\mathcal{D}, N)^{([1],*)} \arrow[ur, "d_0"] \arrow[dr, "d_1"'] & \\ & & (\mathcal{D}, N) \end{tikzcd}$$

A laced functor $f : (\mathcal{C}, M) \to (\mathcal{D}, N)$ is a *trace equivalence* if there exists a laced functor $g : (\mathcal{D}, N) \to (\mathcal{C}, M)$ such that $g \circ f$ and $f \circ g$ are trace homotopic to their respective identities.

■ **Example 5.12** Both $d_0$ and $d_1$ are trace-equivalences, with the same trace-inverse $s : (\mathcal{C}, M) \to (\mathcal{C}, M)^{([1],*)}$. Indeed, since $s \circ d_0 \simeq s \circ d_1 \simeq \mathrm{id}$ already, it suffices to find a laced arrow

$$H : (\mathcal{C}, M)^{([1],*)} \longrightarrow (\mathcal{C}, M)^{([1]\times[1],*)}$$

such that $d_0 \circ H \simeq \mathrm{id}$ and $d_1 \circ H \simeq d_0 \circ s$. We can simply pick $H$ induced by the first projection $[1] \times [1] \to [1]$ via the functoriality of the cotensor. ■

**Remark 5.13** Since $\mathrm{Lace}^{\simeq}$ is the functor corepresented by $\mathrm{Sp}^{\mathrm{fin}}$, there is an equivalence

$$\mathrm{Bar}_{\bullet}\, \underline{\mathrm{Fun}}((\mathcal{C}, M), (\mathcal{D}, N)) \simeq \mathrm{Map}_{\mathbf{TCat}^{\mathrm{Ex}}}\left((\mathcal{C}, M), (\mathcal{D}, N)^{([\bullet],*)}\right)$$

Under this equivalence, the 0-simplices are exactly laced functors $(f, \alpha) : (\mathcal{C}, M) \to (\mathcal{D}, N)$ and 1-simplices are the trace homotopies, whose end points are given by the faces of the simplicial structure.

Let us give a non-trivial example of trace equivalences:

**Lemma 5.14** Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction between exact functors, and let $M : \mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{Sp}$ be exact. Then, the unit $\varepsilon$ and the counit $\eta$ of the adjunction promote $L$ and $R$ to laced functors $L_M := (L, M \circ (\mathrm{id}^{\mathrm{op}} \times \varepsilon))$ and $R_M := (R, M \circ (\eta^{\mathrm{op}} \times \mathrm{id}))$ such that

$$(\mathcal{C}, M \circ (L^{\mathrm{op}} \times \mathrm{id})) \underset{R_M}{\overset{L_M}{\rightleftarrows}} (\mathcal{D}, M \circ (\mathrm{id}^{\mathrm{op}} \times R))$$

are trace-inverses to one another.

**Proof.** What we have described is clearly a pair of laced functors, so it remains to check that both composite are trace equivalent to the identity. The argument will be dual so we focus on furnishing a commutative diagram:

$$\begin{tikzcd}
 & & (\mathcal{C}, M \circ (L^{\mathrm{op}} \times \mathrm{id})) \\
(\mathcal{C}, M \circ (L^{\mathrm{op}} \times \mathrm{id})) \arrow[r, "H"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "R_M \circ L_M"'] & (\mathcal{C}, M \circ (L^{\mathrm{op}} \times \mathrm{id}))^{([1],*)} \arrow[ur, "d_0"] \arrow[dr, "d_1"'] & \\
 & & (\mathcal{C}, M \circ (L^{\mathrm{op}} \times \mathrm{id}))
\end{tikzcd}$$

exhibiting $R_M \circ L_M$ as trace homotopic to id.

On underlying stable categories, we let $H : \mathcal{C} \to \mathcal{C}^{[1]}$ be the functor sending $X$ to $\varepsilon_X : (X \to RL(X))$, and unpacking the definitions via Example 3.31, the natural transformation we have to supply is given on objects $X, Y \in \mathcal{C}$ by:

$$M(L(X), Y) \longrightarrow M(LRL(X), Y)$$

hence we can simply take $M \circ (L(\varepsilon)^{\mathrm{op}} \times \mathrm{id})$. This concludes. □

**Definition 5.15** A functor $F : \mathbf{TCat}^{\mathrm{Ex}} \to \mathcal{E}$ is *trace-like* if it inverts trace equivalences.

**Remark 5.16** Note that since they are pointed, every stable category fits in an adjunction $0 \rightleftarrows \mathcal{C}$. In particular, Lemma 5.14 for $M = 0$ implies that for $F$ *trace-like*, there is an equivalence

$$F(\mathcal{C}, 0) \simeq F(0, 0)$$

Our first result shows a simpler criterion to check that a functor is trace-like.

**Proposition 5.17** Let $F : \mathbf{TCat}^{\mathrm{Ex}} \to \mathcal{E}$ such that any of the two following conditions are met:

- For every laced $(\mathcal{C}, M)$, $F$ sends $d_0, d_1 : (\mathcal{C}, M)^{([1],*)} \to (\mathcal{C}, M)$ to an equivalence.
- For every laced $(\mathcal{C}, M)$, $F$ sends $\widehat{d}_0, \widehat{d}_1 : (\mathcal{C}, M) \to (\mathcal{C}, M)_{([1],*)}$ to an equivalence.

Then, $F$ is trace-like (and in particular, both of the above are realized).

**Proof.** We treat the first case, the second is dual. Let $f : (\mathcal{C}, M) \to (\mathcal{D}, N)$ be a trace-equivalence with trace-inverse $g : (\mathcal{D}, N) \to (\mathcal{C}, M)$, and let $H$ and $H'$ be the two trace-homotopies to the identity. Since $d_0$ is a trace equivalence and $d_0 \circ H \simeq \mathrm{id}$, $F$ maps $H$ to an actual equivalence, and thus, it also maps $d_1 \circ H \simeq g \circ f$ to an equivalence. The same argument applies to $H'$ and shows $F$ also inverts $f \circ g$. Hence $F(f)$ is an actual equivalence with inverse $F(g)$. $\square$

In fact, since there is a diagram

$$\begin{tikzcd}
& & (\mathcal{C}, M) \\
(\mathcal{C}, M) \arrow[r, "s"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "\mathrm{id}"'] & (\mathcal{C}, M)^{([1],*)} \arrow[ur, "d_0"] \arrow[dr, "d_1"] & \\
& & (\mathcal{C}, M)
\end{tikzcd}$$

the first condition of Proposition 5.17 could be stated only for $d_0$ or $d_1$, or for the maps $(\mathcal{C}, M) \to (\mathcal{C}, M)^{([1],*)}$. This also holds dually for $(\mathcal{C}, M)_{([1],*)}$ and $\widehat{s}, \widehat{d}_0, \widehat{d}_1$. In particular, we note that we were a bit vague in Definition 5.11 about which direction the trace homotopies between $g \circ f$ and id or $f \circ g$ and id are supposed to go but the above result implies that all four possible classes of arrows are inverted by trace-like functors, no matter which definition of them is taken.

**Lemma 5.18** Let $(\mathcal{C}, M)$ be a laced category and $[n] \to [m]$ a map in $\Delta$. The following maps are trace-equivalences:

1. $(\mathcal{C}, M)^{([m],*)} \to (\mathcal{C}, M)^{([n],*)}$
2. $(\mathcal{C}, M)_{([n],*)} \to (\mathcal{C}, M)_{([m],*)}$

**Proof.** We treat the first case, the second is dual. Trace equivalences are stable under composition and if $f$ is a trace-equivalence and $g \circ f = \mathrm{id}$ or $f \circ g = \mathrm{id}$, then so is $g$. Hence, given the structure of $\Delta$, it is sufficient to treat the case of the injective map $[n] \to [n+1]$ which only misses $n+1$. Denote $(p, \alpha) : (\mathcal{C}, M)^{([n+1],*)} \to (\mathcal{C}, M)^{([n],*)}$ and $(i, \beta)$ the one-sided inverse given by the projection $[n+1] \to [n]$ sending both $n$ and $n+1$ to $n$. We need to find a trace-homotopy between the composition $ip$ and id: this is a functor $H : (\mathcal{C}, M)^{([n+1],*)} \to (\mathcal{C}, M)^{([n+1]\times[1],*)}$ such that

$$\begin{tikzcd}
& & (\mathcal{C}, M)^{([n+1],*)} \\
(\mathcal{C}, M)^{([n+1],*)} \arrow[r, "H"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "ip"'] & (\mathcal{C}, M)^{([n+1]\times[1],*)} \arrow[ur, "d_0"] \arrow[dr, "d_1"] & \\
& & (\mathcal{C}, M)^{([n+1],*)}
\end{tikzcd}$$

is a commutative diagram. But there is already such a commutative diagram of categories:

$$\begin{tikzcd}
{[n+1]} \arrow[dr, "j_0"] \arrow[drr, bend left, "\mathrm{id}"] & & \\
& {[n+1] \times [1]} \arrow[r, "\tilde{H}"] & {[n+1]} \\
{[n+1]} \arrow[ur, "j_1"] \arrow[urr, bend right, "\tilde{ip}"'] & &
\end{tikzcd}$$

where $\tilde{H} : [n+1] \times [1] \to [n+1]$ maps a tuple $(k, i)$ to $k$ if $i = 0$ or $k \leq n$ and $n$ otherwise. The functoriality of the cotensor then concludes. $\square$

As a consequence of Lemma 5.18, the simplicial object $(\mathcal{C}, M)^{([\bullet],*)}$ has its faces and degeneracies being trace-equivalences; this is similar to how the Q-construction $Q_\bullet(\mathcal{C})$ which is used to define K-theory has its faces being split-Verdier projections, see [CDH+23b, Proposition 2.7.2]. As a consequence of Proposition 5.17, $F$ is trace-like if and only if it sends this simplicial equivalence to the constant simplicial object $F(\mathcal{C}, M)$.

**Definition 5.19** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a functor. We let $\mathrm{cyc}(F)$ denote the functor given pointwise by the following geometric realization:

$$\mathrm{cyc}(F)(\mathcal{C}, M) := \left| F\left( (\mathcal{C}, M)^{([\bullet],*)} \right) \right|$$

The association $F \mapsto \mathrm{cyc}(F)$ upgrades to a functor $\mathrm{cyc} : \mathrm{Fun}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \to \mathrm{Fun}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$, which we call the *cyclic bar construction functor*. The inclusion of 0-simplices provides a natural transformation $\eta_F : F \to \mathrm{cyc}(F)$ which is itself natural in $F$, hence an augmentation $\eta : \mathrm{id} \to \mathrm{cyc}$.

In Proposition 5.10, we have proven that $\mathrm{uTHH} \simeq \mathrm{cyc}(\mathrm{Lace}^{\simeq})$.

**Proposition 5.20** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be any functor. Then $\mathrm{cyc}(F)$ is trace-like.

**Proof.** Using Proposition 5.17, we are reduced to showing that the following simplicial map (actually there are two such maps but they are simplicially homotopic) induces an equivalence on geometric realizations:

$$F((\mathcal{C}, M)^{([\bullet],*)}) \longrightarrow F((\mathcal{C}, M)^{([1]\times[\bullet],*)})$$

The above map has a one-sided inverse, so it suffice to show the composite

$$\phi_\bullet : F((\mathcal{C}, M)^{([1]\times[\bullet],*)}) \longrightarrow F((\mathcal{C}, M)^{([\bullet],*)}) \longrightarrow F((\mathcal{C}, M)^{([1]\times[\bullet],*)})$$

is simplicially homotopic to the identity. This is exactly providing maps $h_i : F((\mathcal{C}, M)^{([1]\times[n],*)}) \to F((\mathcal{C}, M)^{([1]\times[n+1],*)})$ such that $d_0 h_0 = \mathrm{id}$ and $d_n h_n \simeq \phi_n$ (as well as other simplicial relations that we omit to write).

All of the above wanted maps will be induced by maps in $\Delta$. Indeed, we can consider the map $[1] \times [n] \to [1] \times [n+1]$ mapping $(\epsilon, k)$ to itself if $k \geq i$ and to $(0, k)$ otherwise. This satisfies the correct relations, hence the result. $\square$

In particular, we get that uTHH is trace-like. In fact, we now show that it is actually initial with this property under $\mathrm{Lace}^{\simeq}$.

**Theorem 5.21** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a functor. The natural transformation $\eta_F : F \to \mathrm{cyc}(F)$ identifies the latter as the initial trace-like functor under $F$.

**Proof.** We verify that condition (3) of Proposition 5.2.7.4 of [Lur08] is satisfied, since this criterion implies that cyc is left adjoint to the inclusion of trace-like functors into $\mathrm{Fun}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$, with the natural transformation $\eta$ realizing the unit of the adjunction.

We have seen that if $F$ is trace-like, then $\eta_F$ is an equivalence. We now show that for any $F$, the two maps $\eta_{\mathrm{cyc}(F)}$ and $\mathrm{cyc}(\eta_F)$ are homotopic. Remark that

$$((\mathcal{C}, M)^{([n],*)})^{([m],*)} \simeq ((\mathcal{C}, M)^{([m],*)})^{([n],*)} \simeq (\mathcal{C}, M)^{([m]\times[n],*)}$$

hence, $\mathrm{cyc}\,\mathrm{cyc}(F)$ is the colimit of the bisimplicial object $F(((\mathcal{C}, M)^{([\bullet_1]\times[\bullet_2],*)})$, and the two maps $\eta_{\mathrm{cyc}(F)}$ and $\mathrm{cyc}(\eta_F)$ are simply induced by the horizontal and vertical inclusions in $\Delta \times \Delta$. Thus, the flip involution $\Delta \times \Delta \to \Delta \times \Delta$ which reverses the two coordinates yields the wanted homotopy. In particular, since $\mathrm{cyc}(F)$ is trace-like by Proposition 5.20, we see that $\eta_{\mathrm{cyc}(F)}$ is an equivalence and thus, this also holds for $\mathrm{cyc}(\eta_F)$

By combining Proposition 5.20 and Proposition 5.17, we have a functor $\mathrm{cyc} : F \mapsto \mathrm{cyc}(F)$ equipped with a natural transformation $\eta : \mathrm{id} \implies \mathrm{cyc}$ whose image is exactly the full subcategory $\mathcal{T}$ of $\mathrm{Fun}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$ composed of trace-like functors. This is precisely (3) of Proposition 5.2.7.4 of [Lur08] as wanted. □

Combining the Theorem with Proposition 5.10, we have:

**Corollary 5.22** The natural transformation $\mathrm{Lace}^{\simeq} \to \mathrm{uTHH}$ exhibits the latter as the initial trace-like invariant under $\mathrm{Lace}^{\simeq}$. Since $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ commutes with the formation of the cyclic bar construction, this also applies to $\Sigma^\infty_+ \mathrm{Lace}^{\simeq} \to \Sigma^\infty_+ \mathrm{uTHH}$.

## 5.3 Fiberwise-exact invariants and a first universal property for THH

In the previous section, we have shown that the natural transformation $\mathrm{Lace}^{\simeq} \to \mathrm{uTHH}$ identified uTHH as the initial trace-like functor under $\mathrm{Lace}^{\simeq}$. We want to translate this knowledge to the stable THH; for this, we need to understand the natural transformation $\Sigma^\infty_+ \mathrm{uTHH} \to \mathrm{THH}$.

Let us first deal with a simple case: if $F : \mathcal{C} \to \mathcal{D}$ is a functor between differentiable stable categories, then, there is a natural transformation $F \to \mathrm{P}_1 F$ whose target $\mathrm{P}_1 F$ is exact, which is initial among such transformations. We say that $\mathrm{P}_1 F$ is the *exact approximation* of $F$.

**Lemma 5.23** Let $\mathcal{C}$ be a stable category and $F : \mathcal{C} \to \mathrm{Sp}$ an exact functor. The natural transformation $\eta : \Sigma^\infty_+ \Omega^\infty F \to F$ exhibits its target as the initial exact functor under its source.

**Proof.** Recall that the universal property of $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ implies that

$$(\Omega^\infty)^* : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{REx}}(\mathcal{C}, \mathcal{S})$$

is an equivalence. In particular, for every exact $G : \mathcal{C} \to \mathrm{Sp}$, the top horizontal arrow of the following commutative square is also an equivalence:

$$\begin{tikzcd}
\mathrm{Nat}(F, G) \arrow[r, "\simeq"] \arrow[d, "\eta^*"] & \mathrm{Nat}(\Omega^\infty F, \Omega^\infty G) \arrow[d, equal] \\
\mathrm{Nat}(\Sigma^\infty_+ \Omega^\infty F, G) \arrow[r, "\simeq"] & \mathrm{Nat}(\Omega^\infty F, \Omega^\infty G)
\end{tikzcd}$$

Consequently, $\eta^*$ is an equivalence. This concludes. □

In particular, if $\mathcal{C}$ is stable and $X, Y \in \mathcal{C}$, then $\Sigma^\infty_+ \Omega^\infty M(X,Y) \to M(X,Y)$ is an exact approximation in $M$, since $\mathrm{ev}_{X,Y} : M \mapsto M(X,Y)$ is exact. Thus, for a fixed $\mathcal{C}$, the natural transformation

$$\Sigma^\infty_+ \mathrm{uTHH}(\mathcal{C}, -) \implies \mathrm{THH}(\mathcal{C}, -)$$

is obtained by taking a colimit of exact approximations. Since $\mathrm{P}_1$ is a left adjoint it commutes with colimits and it follows that for a fixed $\mathcal{C}$, $\mathrm{THH}(\mathcal{C}, -)$ is the exact approximation of $\Sigma^\infty_+ \mathrm{uTHH}(\mathcal{C}, -)$. To get the universal property when $\mathcal{C}$ varies, we need to introduce a fibered version of the exact approximation. More formally, let us give the following definition:

**Definition 5.24** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a functor, with $\mathcal{E}$ a stable category. We say that $F$ is

- *fiberwise-reduced* if, for every stable $\mathcal{C}$, the restriction $F_{\mathcal{C}} : \mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is reduced, i.e. sends the zero bimodule to zero.
- *fiberwise-exact* if, for every stable $\mathcal{C}$, the restriction $F_{\mathcal{C}} : \mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is exact between stable categories.

In classical Goodwillie calculus, the exact approximation can be realized by first taking the reduced approximation, i.e. splitting away $F(0)$ from $F(X)$ and then forming the excisive approximation, which is given by the following formula when $F$ is reduced[10]: $\mathrm{colim}_n \Omega^n F(\Sigma^n X)$. This idea still works in the bundled version, as we now explain.

[10] and the target category admits filtered colimits, or at the very least, is differentiable in the sense of [Lur17a, Definition 6.1.1.6]

**Lemma 5.25** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a functor to a pointed category. Then, there exists an initial natural transformation $F \to F^{\mathrm{red}}$ whose target is fiberwise-reduced.

Its fiber is the functor $(\mathcal{C}, M) \mapsto F(\mathcal{C}, 0)$; in particular, the retraction $(\mathcal{C}, M) \to (\mathcal{C}, 0)$ induces a splitting:

$$F(\mathcal{C}, M) \simeq F^{\mathrm{red}}(\mathcal{C}, M) \oplus F(\mathcal{C}, 0)$$

for every $(\mathcal{C}, M) \in \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$.

**Proof.** Denote $\mathrm{Fun}_*(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$ the full subcategory of $\mathrm{Fun}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$ spanned by fiberwise-reduced functors, where $\mathcal{E}$ is a pointed category. We want to show the inclusion

$$\mathrm{Fun}_*(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$$

admits a left adjoint. Using the existing literature, we will show that it follows from the existence of a left adjoint for the restrictions to each tangent category $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$ where it is a standard claim that the category of reduced functor is a localization of the category of all functors.

Since $\mathrm{fgt} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Cat}^{\mathrm{Ex}}$ is a cocartesian fibration classifying $\mathrm{Bimod}(-)$, it follows from combining [Lur08, Corollary 3.2.2.13] and [GHN17, Proposition 7.3] that this inclusion can be rewritten

$$\Gamma(\mathrm{Un}^{cart}(\mathrm{Fun}_*(\mathrm{Bimod}(-), \mathcal{E}))) \longrightarrow \Gamma(\mathrm{Un}^{cart}(\mathrm{Fun}(\mathrm{Bimod}(-), \mathcal{E})))$$

where $\mathrm{Un}^{cart}(F)$ denotes the cartesian fibration classifying a functor $F$, and $\Gamma$ denotes the categories of sections of a given fibration.

Applying the dual version of [HY17, Proposition 5.1] for cartesian fibrations, we get that the existence of a left adjoint follows from the existence of a left adjoint for each fiber and that the restriction of the global left adjoint recovers the fiberwise adjoint. The claim and the formula now follow from the case $n = 0$ of [Lur17a, Lemma 6.1.1.33]. ☐

Recall that a pointed category with finite limits and filtered colimits is called differentiable if those two commute.

**Lemma 5.26** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a functor to a differentiable stable $\mathcal{E}$. There exists an initial natural transformation $F \to \mathrm{P}_1^{\mathrm{fbw}} F$ with a fiberwise-exact target such that on each fiber, the restriction of $\mathrm{P}_1^{\mathrm{fbw}} F$ coincides with the exact approximation of the restriction of $F$.

**Proof.** We can play the same yoga as the previous lemma, replacing $\mathrm{Fun}_*(\mathrm{Bimod}(-), \mathcal{E})$ by $\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Bimod}(-), \mathcal{E})$. Again, the pointwise claim and the formula are classical and can be found as the case $n = 1$ of [Lur17a, Lemma 6.1.1.33]. ☐

If $F$ fiberwise-reduced, the usual formula for exact approximations of reduced functors gives:

$$\mathrm{P}_1^{\mathrm{fbw}} F(\mathcal{C}, M) := \operatorname*{colim}_n \Omega^n F(\mathcal{C}, \Sigma^n M) \tag{$\star$}$$

If $F$ is not fiberwise-reduced, the above formula for $\mathrm{P}_1^{\mathrm{fbw}} F$ still holds provided one replaces $F$ by $F^{\mathrm{red}}$ in the right hand side.

**Remark 5.27** Suppose $F$ is trace-like, then $\Omega F(-, \Sigma -)$ is again trace-like. Indeed, for any laced category, we have the following equivalence

$$(\mathcal{C}, \Sigma M)^{([1],*)} \simeq (\mathcal{C}^{([1],*)}, \Sigma M^{([1],*)})$$

as can be easily checked by the explicit formula of Example 3.31. Moreover, we also have a splitting

$$F((\mathcal{C}, M)^{([1],*)}) \simeq F^{\mathrm{red}}((\mathcal{C}, M)^{([1],*)}) \oplus F(\mathcal{C}^{([1],*)}, 0)$$

In particular, it follows that $\mathrm{P}_1^{\mathrm{fbw}}$ preserves trace-like functors using ($\star$).

We can now state the first universal property of THH, first in terms of uTHH and then, using Corollary 5.22, in terms of $\Sigma_+^\infty \mathrm{Lace}^{\simeq}$. We will need the following lemma:

**Proposition 5.28** The natural transformation $\Sigma^\infty_+ \operatorname{uTHH} \to \operatorname{THH}$ exhibits its target as the initial fiberwise-exact functor under its source. In consequence, $\Sigma^\infty_+ \operatorname{Lace}^{\simeq} \to \operatorname{THH}$ realizes THH as the initial fiberwise-exact trace-like invariant under its source.

**Proof.** The second part follows from the first thanks to Remark 5.27, hence we focus on the first part. It suffices to check the formula pointwise by Lemma 5.26 and we have already performed this check after Lemma 5.23. □

**Remark 5.29** The following diagram is a commutative square of fully-faithful right adjoints:

$$\begin{tikzcd} \operatorname{Fun}^{tr-like,fbw-ex}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \arrow[r] \arrow[d] & \operatorname{Fun}^{tr-like}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \arrow[d] \\ \operatorname{Fun}^{fbw-ex}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \arrow[r] & \operatorname{Fun}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E}) \end{tikzcd}$$

It follows that the associated square of localizations obtained by taking left adjoints also commutes. Note that cyc preserve fiberwise-exact functors, which is clear from the formula defining it; moreover $\mathrm{P}_1^{\mathrm{fbw}}$ preserve trace-like functors by Remark 5.27. Consequently, each adjunction must descent along the other localization. Ultimately, this is precisely saying that cyc and $\mathrm{P}_1^{\mathrm{fbw}}$ commute with one another.

Hence, THH can also be obtained as $\operatorname{cyc}(\mathrm{P}_1^{\mathrm{fbw}}\Sigma^\infty_+ \operatorname{Lace}^{\simeq})$, the cyclic bar construction applied to a stabilized version of $\Sigma^\infty_+ \operatorname{Lace}^{\simeq}$; one readily checks that this gives the following formula:

$$\operatorname{THH}(\mathcal{C}, M) \simeq \left| \ \dots \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \operatorname{colim}_{X,Y\in\iota\mathcal{C}} \operatorname{map}_{\mathcal{C}}(X,Y) \otimes M(Y,X) \rightrightarrows \operatorname{colim}_{X\in\iota\mathcal{C}} M(X,X) \right|$$

In particular, for $M = \operatorname{map}_{\mathcal{C}}$, this is what we claimed in Remark 5.5. Remark that the stabilized version of $\Sigma^\infty_+ \operatorname{Lace}^{\simeq}$ is in particular some *naive trace*; by using the colimit formula induced by Lemma 3.13, we deduce the following formula:

$$\mathrm{P}_1^{\mathrm{fbw}}(\Sigma^\infty_+ \operatorname{Lace}^{\simeq})(\mathcal{C}, M) \simeq \operatorname*{colim}_{X\in\iota\mathcal{C}} M(X,X)$$

Note also that its trace-like approximation, THH, is an actual trace: the trace of $M$ seen as an endomorphism of the dualizable $\operatorname{Ind}\mathcal{C}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$, by [HSS17, Proposition 4.5].

As for $\mathrm{K}^{\mathrm{lace}}$, we can upgrade the previous universal property to an absolute statement by adding a lax-monoidality condition:

**Proposition 5.30** The functor $\operatorname{uTHH} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{S}$ upgrades to the initial lax-monoidal trace-like invariant. Consequently, the functor $\operatorname{THH} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ also upgrades to the initial lax-monoidal, fiberwise-exact trace-like invariant.

**Proof.** We begin by proving the claim about uTHH. Denote $\mathcal{T}$ the localisation of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ at trace-equivalences, or equivalently at the collection of arrows $(\mathcal{C}, M)_{([1],*)} \to (\mathcal{C}, M)$. This is not a Bousfield localisation but is nonetheless locally small since $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ is compactly generated by Theorem 3.18 and the second description of the collection of arrows is stable under colimits.

Then, uTHH uniquely factors through the localization $U : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{T}$ and by Proposition 5.22, the resulting functor identifies as $\operatorname{Map}_{\mathcal{T}}(U(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id}), -)$. Hence, the result follows from [Nik16, Corollary 6.8] if we can show this localization is multiplicative. For this, it suffices that the collection of trace-equivalence forms an ideal in arrows of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. This follows from showing that trace-homotopic functors are stable under tensoring with an arbitrary laced category.

Consider a laced functor $H : (\mathcal{C}, M) \to (\mathcal{D}, N)^{([1],*)}$, then there is a composite functor $\overline{H}$ as follows:

$$\begin{tikzcd} \overline{H} : (\mathcal{C}, M) \boxtimes (\mathcal{E}, P) \arrow[r, "H\boxtimes \mathrm{id}_{(\mathcal{E},P)}"] & (\mathcal{D}, N)^{([1],*)} \boxtimes (\mathcal{E}, P) \arrow[r, "\mathrm{id}\boxtimes s_0"] & (\mathcal{D}, N)^{([1],*)} \boxtimes (\mathcal{E}, P)^{([1],*)} \arrow[d, "\eta"] \\ & & ((\mathcal{D}, N) \boxtimes (\mathcal{E}, P))^{([1],*)} \end{tikzcd}$$

Here the last vertical map $\eta$ is obtained by using that $(-)^{([1],*)}$ is lax-monoidal, which follows from the universal property of 3.29 and the fact that $\underline{\mathrm{Fun}}((\mathcal{E},P),-)$ is lax-monoidal as a right adjoint to the obviously monoidal $(\mathcal{E},P)\boxtimes -$. We claim that there is a canonical

$$d_i \circ \overline{H} \simeq (d_i \circ H) \boxtimes \mathrm{id}_{(\mathcal{E},P)}$$

for $i = 0,1$, i.e. that $\overline{H}$ witnesses the homotopy after tensoring by the arbitrary laced category $(\mathcal{E},P)$. This is of course true before taking the last map in the composition defining $\overline{H}$ so that it suffices to remark that there is a commutative diagram

$$\begin{tikzcd}
(\mathcal{D},N)^{([1],*)} \boxtimes (\mathcal{E},P)^{([1],*)} \arrow[r, "\eta"] \arrow[dr, "d_i \boxtimes d_i"'] & ((\mathcal{D},N)\boxtimes(\mathcal{E},P))^{([1],*)} \arrow[d, "d_i"] \\
& (\mathcal{D},N)\boxtimes(\mathcal{E},P)
\end{tikzcd}$$

But this is precisely how the lax-monoidal structure is defined via the adjunction. This concludes for the first part.

We now seek to deduce the claim on THH. Proposition 5.28 reduces us to show that the functor

$$\mathrm{P}_1^{\mathrm{fbw}} : \mathrm{Fun}^{tr-like}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{Sp}) \longrightarrow \mathrm{Fun}^{tr-like, fbw-ex}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{Sp})$$

is monoidal so that it upgrades to a left-adjoint functor between the category of lax-monoidal such functors. But $\mathrm{P}_1^{\mathrm{fbw}}$ is a localization at the collection of arrows $\eta : F(-) \to T_1 F := \Omega F(\Sigma -)$ thus it suffices to check that $\eta \otimes_{Day} G$ is again inverted by the localization for any trace-like $G : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$; in fact, we will not need the trace-like and the result is already true at the level of general functors. Remark that $\eta \otimes_{Day} G$ fits in the following commutative diagram:

$$\begin{tikzcd}
F \otimes_{Day} G \arrow[r, "\eta \otimes_{Day} G"] \arrow[dr, "\eta_{F \otimes_{Day} G}"'] & T_1(F) \otimes_{Day} G \arrow[d] \arrow[dr, "\eta_{T_1(F \otimes_{Day} G)}"] & \\
& T_1(F \otimes_{Day} G) \arrow[r] & T_1(T_1(F) \otimes_{Day} G)
\end{tikzcd}$$

Hence, since arrows inverted by a localization satisfy 2-out-of-6, this concludes. □

## 5.4 Topological Hochschild homology is the derivative of laced K-theory

To realize the result of the title, the first step is the following proposition, which is straightforward from Proposition 5.17.

**Proposition 5.31** Suppose $F$ is fiberwise-reduced and additive. Then $F$ is trace-like.

**Proof.** Using Lemma 4.5, we know that we have a semi-orthogonal decomposition $(\mathcal{C},M)^{([1],*)}$ by $((\mathcal{C},0);(\mathcal{C},M))$ so in particular an exact sequence in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$:

$$(\mathcal{C},0) \longrightarrow (\mathcal{C},M)^{([1],*)} \longrightarrow (\mathcal{C},M)$$

In particular, an additive invariant $F$ sends the above to a fiber sequence, and if it is further reduced, it maps $(\mathcal{C},0)$ to zero, hence the second map becomes an equivalence. By Proposition 5.17, this means $F$ is trace-like. □

Together with the above proposition, Remark 5.16 actually implies that if $F$ is additive, then its fiberwise-reduction coincides with its trace-like approximation. Indeed, additive functors send $0 \in \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ to 0 and thus the trace-like approximation must be fiberwise-reduced; the above also shows that the reduced approximation is automatically trace-like. Therefore we have obtained:

**Corollary 5.32** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be an additive invariant. Then the natural transformation $F \to F^{\mathrm{red}}$, as defined in Lemma 5.25, exhibits its target as the initial trace-like approximation of its source. In other words, there is an equivalence

$$\mathrm{cyc}(F) \simeq F^{\mathrm{red}}$$

under $F$.

We can get a partial converse result by adapting a classical argument, notably found in [Kal15, Section 5.2] who claims it is the gist of the proof of the localization theorem for Hochschild homology proven by [Kel98]. A close variant of this argument also appears in [HSS17, Theorem 3.4]. Note also that we cannot do more than a partial converse, as there are non-additive trace-like functors, even fiberwise-reduced, for instance $(\mathcal{C}, M) \mapsto \mathrm{THH}(\mathcal{C}, M)^{\otimes n}$.

**Theorem 5.33** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a fiberwise-exact functor with target a stable category. Then, the following are equivalent:

(i) $F$ is laced-additive

(ii) $F$ is trace-like

In particular, THH is laced-additive.

**Proof.** Proposition 5.31 gives the first implication since fiberwise-exact functors are in particular fiberwise-reduced. Suppose $F$ is trace-like and exact on each fiber, and let there be an orthogonal decomposition $((\mathcal{A}, N); (\mathcal{B}, P))$ of $(\mathcal{C}, M)$ with $i : \mathcal{A} \to \mathcal{C}$ and $j : \mathcal{B} \to \mathcal{C}$ denoting the inclusions.

Denote $q$ the right adjoint of $i$; Lemma 5.14 applied to the bimodule $M \circ (\mathrm{id}^{\mathrm{op}} \times i)$ and the aforementioned adjunction gives a trace-equivalence

$$(\mathcal{A}, M \circ (i^{\mathrm{op}} \times i)) \longrightarrow (\mathcal{C}, M \circ (\mathrm{id}^{\mathrm{op}} \times iq))$$

whose underlying exact functor is the inclusion. Since $\alpha : N \simeq M \circ (i^{\mathrm{op}} \times i)$, the map $(\mathcal{A}, N) \to (\mathcal{C}, M)$ factors as follows:

$$(\mathcal{A}, N) \longrightarrow (\mathcal{C}, M \circ (\mathrm{id}^{\mathrm{op}} \times iq)) \xrightarrow[(\mathrm{id}, \hat{\alpha})]{} (\mathcal{C}, M)$$

where the first map has been previously described and the natural transformation

$$\hat{\alpha} : M \circ (\mathrm{id}^{\mathrm{op}} \times iq) \implies M$$

is induced by the counit $iq \to \mathrm{id}$ and makes the wanted triangle commute thanks to the triangle identities of the adjunction. We have a dual factorization involving $N \circ (\mathrm{id}^{\mathrm{op}} \times jp)$ where $p$ is the right adjoint to $j$; but since we have a exact sequence $iq \to \mathrm{id} \to jp$ as the underlying sequence is a semi-orthogonal decomposition, this means we have a diagram as follows:

$$\begin{tikzcd}
(\mathcal{A}, N) \arrow[r, "(i,\alpha)"'] \arrow[d, "(i,\mathrm{id})"'] & (\mathcal{C}, N) \arrow[r, "(p,\eta)"] \arrow[d, equal] & (\mathcal{B}, P) \\
(\mathcal{C}, N \circ (\mathrm{id}^{\mathrm{op}} \times iq')) \arrow[r, "(\mathrm{id},\hat{\alpha})"'] & (\mathcal{C}, N) \arrow[r, "(\mathrm{id},\hat{\beta}')"'] & (\mathcal{C}, N \circ (\mathrm{id}^{\mathrm{op}} \times j'p)) \arrow[u, "(p,\mathrm{id})"']
\end{tikzcd}$$

where the bottom horizontal sequence is a (split-)exact sequence of $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$ and the vertical maps are trace equivalences. Since $F$ is exact on each fiber and inverts trace equivalences, it follows that the top row is also sent to an exact sequence, which concludes. $\square$

In particular, we deduce from the above that THH is additive, and actually, the universal fiberwise-exact additive invariant under $\Sigma^{\infty}_{+} \mathrm{Lace}^{\simeq} \to \mathrm{THH}$. From Theorem 4.10, we deduce the following:

**Corollary 5.34 — Stable K-theory is THH.** The canonical natural transformation $\mathrm{K}^{\mathrm{lace}} \to \mathrm{THH}$ exhibits its target as the initial fiberwise-exact functor under the source. In particular, on the fiber over each stable $\mathcal{C}$, the first Goodwillie derivative of the functor $\mathrm{K}(\mathrm{Lace}(\mathcal{C}, -))$ is $\mathrm{THH}(\mathcal{C}, -)$.

This generalizes the main result of [DM94] to stable categories $\mathcal{C}$ which are not $\mathrm{Perf}(R)$ for some connective ring spectrum $R$; but note how we never quite needed any specific constructions of our objects — the only constructions of THH given were to show the well-known construction coincided with the object with the correct universal property, in particular if one were to adopt a definition of THH similar in style to our definition of K-theory, the above is almost purely formal.

We also note that in his thesis [Ram24b], Ramzi has independantly and simultaneously shown a version of above statement, although with yet another proof, which makes use of the category $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$, or $\Lambda^{st}$ in his notations. Further sections of this manuscript will compare the two approaches.

**Remark 5.35** Note that in Proposition 5.30, we have shown generally that $\mathrm{P}_1^{\mathrm{fbw}}$ preserves lax-monoidal functors, hence the canonical natural transformation

$$\mathrm{K}^{\mathrm{lace}} \longrightarrow \mathrm{P}_1^{\mathrm{fbw}}\,\mathrm{K}^{\mathrm{lace}} \simeq \mathrm{THH}$$

is a natural transformation of lax-monoidal functors, i.e. a map of algebra objects in the functor category.

# 6 Higher derivative of laced Verdier-localizing invariants

This section and the next present material related to the paper [HNS25]. This paper is long, technical and has multiple companion papers (or at least, is planned to at the time of writing). We have thus made the following choice of presentation: this section deals with presenting an original exposition of the material, which aims to be shorter but therefore will need to cut corners and avoid dealing with some of the coherence problems. In particular this section does not offer complete, publishable proofs of all the results it states (but it does for many).

In constrast, section 7 is purely a collection of selected useful statements of *loc. cit.*, given without proofs. They all imply the major statements of this section and have their proof in the aforementioned paper. Our hope is that the reader will find in this perilous act of balancing both a legible and a complete picture of the results.

## 6.1 Fiberwise Goodwillie calculus

In previous sections, we have shown that for every functor $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$, where $\mathcal{E}$ is stable with sequential colimits, there exists an initial $F \to \mathrm{P}_1^{\mathrm{fbw}} F$ with target a functor such that for each $\mathcal{C}$, $\mathrm{P}_1^{\mathrm{fbw}} F(\mathcal{C}, -)$ is exact. We coined the term *fiberwise-exact* functors. Moreover, if $F$ is *fiberwise-reduced*, i.e. $F(\mathcal{C}, 0) \simeq 0$ for every $\mathcal{C}$, then we have the usual classical formula:

$$\mathrm{P}_1^{\mathrm{fbw}} F(\mathcal{C}, M) \simeq \operatorname*{colim}_n \Omega^n F(\mathcal{C}, \Sigma^n M)$$

We want now to introduce the higher polynomial approximations in the fiberwise setting. Recall from Chapter 6 of [Lur17a] the following notions of Goodwillie calculus: a strongly cocartesian $n$-cube is a functor $X : \mathcal{P}([n]) \to \mathcal{C}$, where $\mathcal{P}([n])$ is the poset of subsets of $[n]$ ordered by inclusion, which is left Kan extended from its restriction to subsets of size $\leq 1$. A reduced functor is said to be $n$-excisive if it carries strongly cocartesian $n$-cubes to cartesian cubes.

We will adopt the following terminology: a fiberwise strongly cocartesian $n$-cube is a strongly cocartesian cube with target $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$ for some stable $\mathcal{C}$.

**Definition 6.1** Let $n \in \mathbb{N}^*$, a functor $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is said to be *fiberwise $n$-excisive* if for every $\mathcal{C}$, $F(\mathcal{C}, -) : \mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is $n$-excisive, i.e. sends strongly cocartesian $n$-cubes to cartesian cubes.

In ordinary Goodwillie calculus, [Lur17a, Lemma 6.1.1.33] guarantees that for every functor $F : \mathcal{C} \to \mathcal{D}$ to a stable category with sequential colimits, there is an initial natural transformation $F \to \mathrm{P}_n F$ whose target is $n$-excisive and source is $F$. We have a similar fiberwise result:

**Theorem 6.2** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a functor to a stable category with sequential colimits. Then, there exists an initial natural transformation $F \to \mathrm{P}_n^{\mathrm{fbw}} F$ with target a fiberwise $n$-excisive functor. Moreover, if $F_{\mathcal{C}}$ denotes the restriction of $F$ to some fiber $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}}$, then

$$\mathrm{P}_n^{\mathrm{fbw}} F(\mathcal{C}, M) \simeq \mathrm{P}_n(F_{\mathcal{C}})(M)$$

where $\mathrm{P}_n$ denotes the usual $n$-excisive approximation.

**Proof.** We argue just as in the Lemmas 5.25 and 5.26, using [Lur17a, Lemma 6.1.1.33] for the existence on each fiber. □

As for ordinary Goodwillie calculus, we can define fiberwise $n$-homogenenous functors as those $n$-excisive functors $F$ such that $\mathrm{P}_{n-1}^{\mathrm{fbw}} F \simeq 0$. The fiberwise $n$-homogeneous part $\mathrm{D}_n^{\mathrm{fbw}} F$ of a functor $F$ is then given by the fiber of the natural transformation $\mathrm{P}_n^{\mathrm{fbw}} F \to \mathrm{P}_{n-1}^{\mathrm{fbw}} F$. It is in particular $n$-homogeneous because $\mathrm{P}_{n-1}^{\mathrm{fbw}}$ commutes with fibers. The above theorem implies that

$$\mathrm{D}_n^{\mathrm{fbw}} F(\mathcal{C}, M) \simeq \mathrm{D}_n(F_{\mathcal{C}})(M)$$

where $\mathrm{D}_n(G) := \mathrm{fib}(\mathrm{P}_n G \to \mathrm{P}_{n-1} G)$. Recall that in ordinary Goodwillie calculus, $\mathrm{D}_n(G)$ can be computed via the $n^{th}$-cross effect. Namely, if $G : \mathcal{C} \to \mathcal{D}$ is a functor, then $\mathrm{cr}_n(G)$ is the initial functor $\mathcal{C}^{\times n} \to \mathcal{D}$ receiving a natural transformation from $G(X_1 \oplus ... X_n)$ which is reduced in each variable.

The map $G(X_1 \oplus ... X_n) \to \mathrm{cr}_n(G)(X_1, ..., X_n)$ is a split-projection; note also that since $G$ is symmetric in all the variables, the $n^{th}$-cross effect factors through $(\mathcal{C}^{\times n})^{\mathrm{h}\Sigma_n}$. We write $\mathrm{cr}_{(n)} : (\mathcal{C}^{\times n})^{\mathrm{h}\Sigma_n} \to \mathcal{D}$ for the induced functor.

For such functors in many variables, we can successively in each variable (and thanks to the symmetry, independantly of the order) linearize. If $G : \mathcal{C}^{\times n} \to \mathcal{D}$ is any functor, there is an initial

$$G \longrightarrow \mathrm{P}_{1,\dots,1} G$$

whose target is 1-excisive in every variable. In fact, more generally if $\vec{m} := (m_1, ..., m_n)$ is a tuple of natural integers, we can consider $\mathrm{P}_{\vec{m}} G$ which receives the initial natural transformation out of $G$ which is $m_i$-excisive in the variable $i$. In [Lur17a, Theorem 6.1.4.7, Proposition 6.1.4.14], Lurie shows that

$$(\mathrm{P}_{1,\dots,1} \mathrm{cr}_{(n)} G)(X, ..., X)_{\mathrm{h}\Sigma_n} \simeq \mathrm{D}_n(G)(X)$$

a result known as the classification of $n$-homogenenous functors. Here the $\Sigma_n$-action on the diagonal is a consequence of the factorization through $(\mathcal{C}^{\times n})^{\mathrm{h}\Sigma_n}$.

In our setting, this idea still applies and gives rises to the following result:

**Corollary 6.3** Let $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a fiberwise-reduced functor. Then, we have an equivalence

$$\mathrm{D}_n^{\mathrm{fbw}} F(\mathcal{C}, M) \simeq (\mathrm{P}_{1,\dots,1} \mathrm{cr}_{(n)} F_{\mathcal{C}})(M, ..., M)_{\mathrm{h}\Sigma_n}$$

where $\mathrm{cr}_{(n)}$ denotes the (ordinary) $n^{th}$-cross effect and $\mathrm{P}_{1,\dots,1}$ the first approximation in each of the $n$ variables.

## 6.2 Cyclic invariance in the tangent bundle

The goal of this section is to build towards computation of higher derivatives of $\mathrm{K}^{\mathrm{lace}}$. Note that, just like the next part, we will not end up using material of this section for latter purposes, but we present here as a way to show what can be done by staying within $\mathbf{TCat}^{\mathrm{Ex}}$.

The main obstruction to implementing trace-invariance is the fact that given $M$ a $(\mathcal{D}, \mathcal{C})$-bimodule and $N$ a a $(\mathcal{C}, \mathcal{D})$-bimodule, there is no laced functor between $(\mathcal{C}, N \otimes_{\mathcal{D}} M)$ and $(\mathcal{D}, M \otimes_{\mathcal{C}} N)$. To circumvent this issue, we will build an intermediate laced category which maps to both. Its underlying stable category is the following:

**Definition 6.4** Let $\mathcal{C}, \mathcal{D}$ be stable categories and $M$ a $(\mathcal{D}, \mathcal{C})$-bimodule. We denote $\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M)$ the category $\mathrm{Lace}(\mathcal{C} \times \mathcal{D}, 0 \times M)$, i.e. the category given by the lax-equalizer of $j_{\mathcal{C} \times \mathcal{D}}$ and $0 \times M$, where $M$ is viewed as a functor $\mathcal{C} \to \mathrm{Ind}\, \mathcal{D}$.

There are two functors $p_{\mathcal{C}} : \mathrm{Pair}(\mathcal{C}, \mathcal{D}, M) \to \mathcal{C}$ and $p_{\mathcal{D}} : \mathrm{Pair}(\mathcal{C}, \mathcal{D}, M) \to \mathcal{D}$. We want to refine those to laced functors. For this, let us first investigate what bimodules on $\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M)$.

**Lemma 6.5** The category $\mathrm{Ind}\,\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M)$ is the lax-equalizer of the endofunctors of $\mathrm{Ind}\,\mathcal{C} \times \mathrm{Ind}\,\mathcal{D}$ given by id and $0 \times M$.

**Proof.** Remark that the projection $\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M) \to \mathcal{C}$ admits a left adjoint, given by sending $X$ to $(X, 0, 0 : 0 \to M(X))$. This left adjoint is fully-faithful and every object $(X, Y, f : Y \to M(X))$ therefore fits in an exact sequence

$$(X, 0, 0) \longrightarrow (X, Y, f : Y \to M(X)) \longrightarrow (0, Y, 0 : Y \to 0)$$

The second term $(0, Y, 0 : Y \to 0)$ is the image of the fully-faithful right adjoint of the other projection $\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M) \to \mathcal{D}$. In particular, we see that $\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M)$ is a semi-orthogonal decomposition of $\mathcal{C}$ and $\mathcal{D}$.

Therefore, we get that $\mathrm{Ind}\,\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M)$ is a stable recollement of $\mathrm{Ind}\,\mathcal{C}$ and $\mathrm{Ind}\,\mathcal{D}$ which is classified by the gluing functor $\mathrm{Ind}(M) : \mathrm{Ind}\,\mathcal{C} \to \mathrm{Ind}\,\mathcal{D}$. It now suffices to check that the claimed lax-equalizer is also this stable recollement, which follows from the previous argument *mutatis mutandis* combined with the remark that the existence of an extra (right) adjoint for the projection to $\mathrm{Ind}\,\mathcal{C}$ is a consequence of the stability of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ex}}$ under pullbacks of categories. □

**Remark 6.6** The similar statement does not hold for Lace: it is not true that $\mathrm{Ind}\,\mathrm{Lace}$ is the lax-equalizer of id and $M : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$. Here it is essential that the bimodule is of the form $0 \times M$.

**Proposition 6.7** If $\mathcal{C}, \mathcal{D}$ are stable categories $M$ a $(\mathcal{D}, \mathcal{C})$-bimodule and $N$ a $(\mathcal{C}, \mathcal{D})$-bimodule, there is a laced category $(\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M), \hat{N})$ where $\hat{N}$ is the $\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M)$-bimodule given in formula by

$$\begin{aligned} \mathrm{Pair}(\mathcal{C}, \mathcal{D}, M) &\longrightarrow \mathrm{Ind}\,\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M) \\ (X, Y, Y \to M(X)) &\longmapsto (N(Y), MN(Y), \mathrm{id}) \end{aligned}$$

**Proof.** We claim there is a well-defined exact functor $\mathcal{D} \to \mathrm{Ind}\,\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M)$ which gives the wanted $\hat{N}$ by precomposing by $p_{\mathcal{D}}$. By Lemma 6.5, it suffices to produce a commutative diagram of exact functors

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathrm{Ar}(\mathrm{Ind}\,\mathcal{C}) \times \mathrm{Ar}(\mathrm{Ind}\,\mathcal{D}) \arrow[d] \\ \mathrm{Ind}\,\mathcal{C} \times \mathrm{Ind}\,\mathcal{D} \arrow[r, "(j, 0 \times M)"] & (\mathrm{Ind}\,\mathcal{C} \times \mathrm{Ind}\,\mathcal{D})^2 \end{tikzcd}$$

The vertical left functor is given by the pair $(N, M \circ N)$ whereas the horizontal top functor is given by $(0, \mathrm{id}_{M \circ N})$. The diagram is easily checked to commute, and this gives $\hat{N}$ which satisfies the wanted formula. □

**Proposition 6.8** There is a laced functor

$$(p_{\mathcal{D}}, \alpha) : (\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M), \hat{N}) \longrightarrow (\mathcal{D}, M \circ N)$$

Moreover, this functor is a trace equivalence.

**Proof.** Given the explicit description of $\hat{N}$, we can take the natural transformation $\alpha$ to be the identity on the $\mathcal{D}$-component (and necessarily zero elsewhere), and we have built the underlying functor of stable categories already.

For the second part, remark that $\mathrm{Ind}(p_{\mathcal{D}})\hat{N}L_{\mathcal{D}} \simeq M \circ N$ whereas $\hat{N} \simeq \hat{N}L_{\mathcal{D}}p_{\mathcal{D}}$ hence if we let $P := \hat{N}L_{\mathcal{D}}$, the situation is exactly (up to translating to the one variable context) that of Lemma 5.14. □

In general, however, it does not hold that the other laced functor $(\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M), \hat{N}) \longrightarrow$

$(\mathcal{C}, N \circ M)$ is a trace equivalence, and the situation is as follows:

$$\begin{tikzcd}[column sep=small] & (\mathrm{Pair}(\mathcal{C},\mathcal{D},M),\hat{N}) \arrow[dl] \arrow[dr, "\simeq"] & \\ (\mathcal{C}, N \circ M) & & (\mathcal{D}, M \circ N) \\ & (\mathrm{Pair}(\mathcal{D},\mathcal{C},N),\hat{M}) \arrow[ul, "\simeq"] \arrow[ur] & \end{tikzcd} \tag{$\star$}$$

where we labeled the edges which are trace-equivalences (the other two are not).

Of course, there are more complicated diagrams that appear for cyclic graphs of order $n \geq 2$ of categories and bimodules between them, and those are necessary for the higher coherence data. Since the goal of this section of the manuscript is not to redo what is done in the fully-coherent manner in [HNS25], we will never write those higher diagrams and only suggest them. In particular, we stress that the arguments of this section should be thought of as explaining what is happening under the hood in *loc. cit.* than trying to prove those results.

**Definition 6.9** A functor $F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ is said to be *invariant under cyclic permutations* if it sends every arrow of the above square to an equivalence, and the higher coherence diagrams we did not write as well.

If $F$ is invariant under cyclic permutations, there are two equivalences $F(\mathcal{C}, N \circ M) \xrightarrow{\simeq} F(\mathcal{D}, M \circ N)$. In particular, if $M$ is a $\mathcal{C}$-bimodule, then $F(\mathcal{C}, M \circ M)$ admits a $\mathrm{C}_2$-action and more generally, $F(\mathcal{C}, M^{\otimes n})$ admits an action by $\mathrm{C}_n$, the cyclic group with $n$ elements.

We now seek to exhibit trace-like functors which are invariant under cyclic permutations. For those, it remains to study the other leg of the span, which we do now.

**Lemma 6.10** There is an equivalence $\mathrm{Lace}(\mathrm{Pair}(\mathcal{C},\mathcal{D},M),\hat{N}) \simeq \mathrm{Lace}(\mathcal{C} \times \mathcal{D}, M \times N)$. Moreover, the induced functor

$$\mathrm{Lace}(\mathcal{C} \times \mathcal{D}, M \times N) \to \mathrm{Lace}(\mathcal{C}, M \circ N)$$

is a Verdier projection with kernel $\mathcal{D}$.

**Proof.** The equivalence of Lace categories amounts to the following observation: objects in the left hand side are of the form $(X, Y, f : Y \to M(X))$ and maps $(g : X \to N(Y), h : Y \to MN(Y))$ as well as a homotopy showing that $h \simeq g \circ f$, but the datum of both $h$ and such a homotopy is canonical up to a contractible choice, so it can be removed without any loss.

For the second part, consider the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Lace}(\mathcal{C} \times \mathcal{D}, M \times N) \arrow[r, "p_{\mathcal{C}}"] \arrow[dl, "p_{\mathcal{D}}"'] \arrow[d] & \mathrm{Lace}(\mathcal{C}, M \circ N) \arrow[d] \\ \mathrm{Lace}(\mathcal{D}, N \circ M) \arrow[d] & \mathrm{Lace}(\mathrm{Ind}\,\mathcal{C} \times \mathcal{D}, M \times N) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Lace}(\mathrm{Ind}\,\mathcal{C}, M \circ N) \\ \mathrm{Lace}(\mathrm{Ind}\,\mathcal{D}, N \circ M) & \mathrm{Lace}(\mathrm{Ind}\,\mathcal{C} \times \mathrm{Ind}\,\mathcal{D}, M \times N) \arrow[l, "q"] \arrow[ur] & \end{tikzcd}$$

where we indiscriminately wrote $M$ for the exact functor $\mathcal{C} \to \mathrm{Ind}\,\mathcal{D}$ or $\mathrm{Ind}\,\mathcal{C} \to \mathrm{Ind}\,\mathcal{D}$ (and same for $N$). It is easy to check that both squares are pullback squares and that every vertical map is fully-faithful. Moreover, one can check that the right adjoint to $q$ is given in formula by

$$(Y, f : X \to NM(Y)) \longmapsto (N(Y), Y, \mathrm{id}_{N(Y)}, f)$$

This right adjoint is fully-faithful: the hard part is checking that any map $N(Y) \to N(Y')$ comes from the map from $Y \to Y'$, but this follows from the commutative diagram which involves $\mathrm{id}_{N(Y)}$ and $\mathrm{id}_{N(Y')}$, also provided as part of the maps. In particular, $q$ is a Verdier projection. Since Verdier projections are stable under pullbacks, the functor

$$\mathrm{Lace}(\mathrm{Ind}\,\mathcal{C} \times \mathcal{D}, M \times N) \to \mathrm{Lace}(\mathcal{D}, N \circ M)$$

is also a Verdier projection. Its fiber is the full subcategory of $\mathrm{Lace}(\mathrm{Ind}\,\mathcal{C}\times\mathcal{D}, M\times N)$ spanned by tuples $(X, Y, Y\to M(X), X\to N(Y))$ where $Y\simeq 0$ and the composite $Y\to NM(Y)$ is nullhomotopic. Of course, the second condition is implied by the first which also implies that both $Y\to M(X)$ and $X\to N(Y)$ are zero maps; hence the subcategory in question is equivalent to $\mathrm{Ind}\,\mathcal{C}$, so that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Lace}(\mathcal{C}\times\mathcal{D}, M\times N) \arrow[r] \arrow[d] & \mathrm{Lace}(\mathcal{D}, N\circ M) \arrow[d, equal] \\
\mathrm{Ind}\,\mathcal{C} \arrow[r] & \mathrm{Lace}(\mathrm{Ind}\,\mathcal{C}\times\mathcal{D}, M\times N) \arrow[r] & \mathrm{Lace}(\mathcal{D}, N\circ M)
\end{tikzcd}$$

where the bottom sequence is a fiber-cofiber sequence. The same argument as above shows that the top sequence is a fiber sequence, and clearly, the left hand square is a pullback square. In particular, this implies that the cofiber of the top map in $\mathbf{Cat}^{\mathrm{Ex}}$ is some full subcategory of $\mathrm{Lace}(\mathcal{D}, N\circ M)$ which is characterized as being the essential image of the composite

$$\mathrm{Lace}(\mathcal{C}\times\mathcal{D}, M\times N)\to\mathrm{Lace}(\mathrm{Ind}\,\mathcal{C}\times\mathcal{D}, M\times N)\to\mathrm{Lace}(\mathcal{D}, N\circ M)$$

This composite is essentially surjective, hence the wanted map is a Verdier projection. □

**Remark 6.11** In fact, we could have shown slightly more: the laced functor dual to the one of Proposition 6.8

$$(p_{\mathcal{C}}, \beta) : (\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M), \hat{N}) \longrightarrow (\mathcal{C}, M\circ N)$$

is a fine Verdier projection as in Definition 4.15 whose kernel is $(\mathcal{D}, 0)$. In the previous Lemma, we have only dealt with the condition on Lace (technically for $n=1$, but the cases $n\geq 2$ also follow) and to get the full claim, one need to check that $p\,\mathrm{Pair}(\mathcal{C}, \mathcal{D}, M)\to\mathcal{C}$ is a localization with kernel $\mathcal{D}$ (this already features in the proof of Lemma 6.5), that $\hat{N}$ vanishes when restricted to this kernel and that $\beta$ exhibits $M\circ N$ as left Kan extended from $\hat{N}$ along $p_{\mathcal{C}}$.

Since the proof is invariant under switching the roles of $M$ and $N$, this means that every fiberwise-reduced, weakly-laced Verdier localizing invariant is also invariant under cyclic permutations. As in Theorem 5.33, the converse will holds under the stronger assumption that the functors in question are fiberwise-exact.

Combining the previous lemma with Proposition 6.8, we get many examples of functors invariant under cyclic permutations. Let us first introduce a bit of notation:

**Definition 6.12** Let $F : \mathbf{Cat}^{\mathrm{Ex}}\to\mathcal{E}$ be an additive invariant. We write $F^{\mathrm{cyc}}$ for the (split) fiber of the map $F(\mathrm{Lace}(\mathcal{C}, M))\to F(\mathcal{C})$.

In particular, $F^{\mathrm{cyc}}$ coincides with $(F^{\mathrm{lace}})^{\mathrm{red}}$, where the superscript was introduced in Lemma 5.25. Since $F^{\mathrm{cyc}}$ is also additive, it is trace-like and by Corollary 5.32, we also have $F^{\mathrm{cyc}}\simeq\mathrm{cyc}(F^{\mathrm{lace}})$. We now argue that there is more trace-invariance when $F$ is Verdier-localizing.

**Theorem 6.13** Let $F : \mathbf{Cat}^{\mathrm{Ex}}\to\mathcal{E}$ be a Verdier-localizing invariant. Then, $F^{\mathrm{cyc}}$ and $\mathrm{P}_1^{\mathrm{fbw}}F^{\mathrm{cyc}}$ are invariant under cyclic permutations.

**Proof.** Since $F^{\mathrm{cyc}}$ and $\mathrm{P}_1^{\mathrm{fbw}}F^{\mathrm{cyc}}$ additive and fiberwise-reduced, it is also trace-like by Proposition 5.31. Therefore we only focus on the non-labeled arrows in $(\star)$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d, equal] & \mathrm{Lace}(\mathcal{C}\times\mathcal{D}, M\times N) \arrow[r] \arrow[d] & \mathrm{Lace}(\mathcal{C}, M\circ N) \arrow[d] \\
\mathcal{D} \arrow[r] & \mathcal{C}\times\mathcal{D} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

with rows Verdier sequences by Lemma 6.10. Applying $F$ and then taking vertical fibers implies that $F^{\mathrm{cyc}}$ sends the non-labeled maps in the square $(\star)$ to equivalences. Now notice that the square $(\star)$ is stable under suspension in the bimodule coordinate, i.e. the operation of $F\mapsto F(-, \Sigma(-))$ shifting in the fiber preserves functors invariant under cyclic permutations. The claim about $\mathrm{P}_1^{\mathrm{fbw}}$ now readily follows from the explicit formula recalled at the beginning of section 6. □

In particular, since K-theory is Verdier-localizing and the first derivative of laced K-theory is THH by Theorem 5.34, we get immediately:

**Corollary 6.14** Cyclic K-theory $\mathrm{K}^{\mathrm{cyc}}$, the (split) fiber of the map $\mathrm{K}^{\mathrm{lace}}(\mathcal{C}, M) \to \mathrm{K}(\mathcal{C})$, is invariant under cyclic permutations.

**Corollary 6.15** The functor $\mathrm{THH} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is invariant under cyclic permutations.

### 6.3 A low-tech computation of homogeneous parts

In this section, we present the furthest we are able to go in the trace-method story while staying in the confines of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. We will not need the results in this section, as they will be superseded by further development but we thought worthwhile to include it, as it presents some of the results in a less technical way.

The goal of this section is to compute $\mathrm{D}_n^{\mathrm{fbw}}\,\mathrm{K}^{\mathrm{lace}}$ and more generally, $\mathrm{D}_n^{\mathrm{fbw}} F^{\mathrm{lace}}$ the fiberwise $n$-homogeneous part of a laced invariant. For this, we want to use Corollary 6.3 and identify instead the derivative in each variable of the $n$-cross effect. Let us begin by the following lemma:

**Lemma 6.16** Let $\mathcal{C}$ be a stable category and $M, N$ two $\mathcal{C}$-bimodules. There is an equivalence

$$\mathrm{Lace}(\mathcal{C}, M \oplus N) \simeq \mathrm{Lace}(\mathrm{Lace}(\mathcal{C}, M), \tilde{N})$$

where $\tilde{N}$ is the $\mathrm{Lace}(\mathcal{C}, M)$-bimodule obtained by pulling $N$ along $p : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}$.

**Proof.** Let $\mathcal{C}$ be a stable category and $M, N$ two $\mathcal{C}$-bimodules. We claim the canonical map $p : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}$ upgrades to a laced functor

$$(\mathrm{Lace}(\mathcal{C}, M), \tilde{N}) \to (\mathcal{C}, M \oplus N)$$

For this, it suffices to supply a natural transformation

$$\tilde{N} \simeq N \circ (p^{\mathrm{op}} \times p) \longrightarrow (M \oplus N) \circ (p^{\mathrm{op}} \times p)$$

and using that composition distributes, we take the canonical map of the coproduct.

A point in $\mathrm{Lace}(\mathcal{C}, M \oplus N)$ is the datum of an object $X \in \mathcal{C}$ and two maps $\mathbb{S} \to M(X, X)$ and $\mathbb{S} \to N(X, X)$. Of course, we can repackage this as an object $\hat{X} := (X, \mathbb{S} \to M(X, X))$ in $\mathrm{Lace}(\mathcal{C}, M)$ and a map $\mathbb{S} \to \tilde{N}(\hat{X}, \hat{X})$ where $\tilde{N}(\hat{X}, \hat{X}) := N(X, X)$. Writing this argument more properly shows that the laced functor we built induces an equivalence on Lace, which concludes. $\square$

We will also need the following statement, a weaker version we already proved in Lemma 5.23.

**Lemma 6.17** Let $\mathcal{C}$ be a pointed, presentable category. Then, the functor $\Sigma^\infty\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ has first Goodwillie derivative $\mathrm{id}_{\mathrm{Sp}(\mathcal{C})}$.

**Proof.** The counit of the adjunction furnishes a map $\eta : \Sigma^\infty\Omega^\infty \to \mathrm{id}$, and for every exact $F$, a commutative square

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{id}, F) \arrow[r, "\Omega^\infty"] \arrow[d, "\eta^*"] & \mathrm{Nat}(\Omega^\infty, \Omega^\infty F) \arrow[d, "="] \\
\mathrm{Nat}(\Sigma^\infty\Omega^\infty, F) \arrow[r, "\simeq"] & \mathrm{Nat}(\Omega^\infty, \Omega^\infty F)
\end{tikzcd}$$

The top horizontal arrow is an equivalence since $(\Omega^\infty)_*$ is fully-faithful as a functor with source $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{D}, \mathrm{Sp}(\mathcal{C}))$, hence so is $\eta$. $\square$

If $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is a Verdier-localizing invariant, then, $F^{\mathrm{cyc}}$ is also invariant under cyclic permutations by Theorem 6.13. In consequence, there is a map

$$F^{\mathrm{cyc}}(\mathcal{C}, M) \longrightarrow F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{hC}_n}$$

Indeed, we have a $\mathrm{C}_n$-equivariant diagonal map $(\mathcal{C}, M) \to (\mathcal{C}^n, M^n)$ in $\mathbf{TCat}^{\mathrm{Ex}}$ and by 6.10, $F^{\mathrm{cyc}}$ applied to the latter is equivalent to $F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})$ and this is a $\mathrm{C}_n$-equivariant equivalence.

It follows that the above induces a map

$$F^{\mathrm{cyc}}(\mathcal{C}, M) \longrightarrow \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{hC}_n}$$

but the right hand side is $n$-excisive, so this map factors through the $n^{th}$-derivative. In fact, we have a commutative square as follows:

$$\begin{tikzcd} \mathrm{P}_n^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M) \arrow[r] \arrow[d] & \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{hC}_n} \arrow[d] \\ \mathrm{P}_{n-1}^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M) \arrow[r] & \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{tC}_n} \end{tikzcd} \tag{KMc$\square$}$$

where the bottom vertical map follows from the fact that $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{tC}_n}$ is at worst $(n-1)$-excisive. Regarding this last fact, let us indicate that in Lemma 6.22, we will prove in fact that if $n$ is prime, then this functor is 1-excisive; the general proof can be extracted from the same observations.

The main result of this section is the fact that when $F$ is finitary i.e. commutes with filtered colimits, this square is cartesian. In fact, we will only need that the derivative $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}$ commutes with fiberwise filtered colimits.

We view this result as some version of [McC01, Proposition 4] or [Kuh04, Lemma 5.2, Proposition 1.9], what we call the Kuhn-McCarthy square of Goodwillie calculus, which is the lesser known counterpart of say [Lur17a, Proposition 6.1.4.14] which only computes the common fiber of the square.

**Theorem 6.18** Let $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ be a finitary Verdier-localizing invariant. Then, the square (KMc$\square$) is cartesian. In particular, the $n$-homogeneous part of $F$ is given by

$$(\mathcal{C}, M) \mapsto \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})_{\mathrm{hC}_n}$$

**Sketch of a proof.** We will not prove this result completely, though we will get tantalizingly close. This theorem is nonetheless true and proven completely in the paper [HNS25]; the part of the proof we are missing is the one which pieces together the coherences of many identifications: this requires significantly more technology than is reasonable to develop and it also makes the argument harder to believe (at least to the crowd of mathematicians this write-up is intended for), because most of the space would be occupied by those coherence questions. Hence, we propose here an incomplete argument, which gives at least a good reason to believe in the statement.

Let us now tackle the mathematical content. Since the target category of $F$ is stable, it suffices to show that the induced map on fibers

$$\mathrm{D}_n^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M) \to \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M)_{\mathrm{hC}_n}$$

is an equivalence. Using [Lur17a, 6.1.4.7], it suffices to check that the first derivative in each variable of the $n$-cross effect of the above map is an equivalence.

On the right-hand side, a standard computation shows this is the $\Sigma_n$-object freely induced from the $\mathrm{C}_n$-object $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M_1 \otimes \ldots \otimes M_n)$; in symbols, the following:

$$\bigoplus_{[f] \in \Sigma_n / \mathrm{C}_n} \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M_{f(1)} \otimes \ldots \otimes M_{f(n)})$$

where $\Sigma_n$ acts by permutation of summands (i.e. by composition on $f$) for the $\Sigma_n/\mathrm{C}_n$-part and through the $\mathrm{C}_n$-action on each summand.

By Lemma 6.16, we have $\mathrm{Lace}(\mathcal{C}, M \oplus N) \xrightarrow{\simeq} \mathrm{Lace}(\mathrm{Lace}(\mathcal{C}, M), \widetilde{N})$ where $\widetilde{N}$ stands for the precomposition $N \circ (p_M^{\mathrm{op}} \times p_M)$ with $p_M : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}$. This means we have the following equivalence:

$$\begin{aligned} F^{\text{lace}}(\mathcal{C}, M \oplus N) &\xrightarrow{\simeq} F^{\text{lace}}(\text{Lace}(\mathcal{C}, M), \widetilde{N}) \\ &\xrightarrow{\simeq} F^{\text{lace}}(\text{Lace}(\mathcal{C}, M), 0) \oplus F^{\text{cyc}}(\text{Lace}(\mathcal{C}, M), \widetilde{N}) \\ &\xrightarrow{\simeq} F^{\text{lace}}(\mathcal{C}, M) \oplus F^{\text{cyc}}(\text{Lace}(\mathcal{C}, M), \widetilde{N}) \end{aligned}$$

In particular, $F^{\text{cyc}}(\mathcal{C}, M \oplus N) \xrightarrow{\simeq} F^{\text{cyc}}(\mathcal{C}, M) \oplus F^{\text{cyc}}(\text{Lace}(\mathcal{C}, M), \widetilde{N})$. We deduce the following natural splitting:

$$F^{\text{cyc}}(\mathcal{C}, M_1 \oplus ... \oplus M_n) \xrightarrow{\simeq} F^{\text{cyc}}(\mathcal{C}, M_1 \oplus .. \oplus M_{n-1}) \oplus F^{\text{cyc}}(\text{Lace}(\mathcal{C}, M_1 \oplus .. \oplus M_{n-1}), \widetilde{M_n})$$

whose supplementary summand is exactly the reduced cross-effect in the variable $M_n$. Since $\widetilde{M_n}$ is linear in $M_n$, we get that the derivative of the above in said variable is

$$\mathrm{P}_1^{\text{fbw}} F^{\text{cyc}}(\text{Lace}(\mathcal{C}, M_1 \oplus ... \oplus M_{n-1}), \widetilde{M_n})$$

At this point, we are thus reduced to show that there is a map

$$\mathrm{P}_1^{\text{fbw}} F^{\text{cyc}}(\text{Lace}(\mathcal{C}, M_1 \oplus ... \oplus M_{n-1}), \widetilde{M_n}) \to \bigoplus_{[f] \in \Sigma_n / \mathrm{C}_n} \mathrm{P}_1^{\text{fbw}} F^{\text{cyc}}(\mathcal{C}, M_{f(1)} \otimes ... \otimes M_{f(n)})$$

which exhibits its target as the first Goodwillie derivative with respect to $M_1, ..., M_{n-1}$ of the source. We now crucially use the fact that $\mathrm{P}_1^{\text{fbw}} F^{\text{cyc}}$ is invariant under cyclic permutations, which we proved in Theorem 6.13.

Remark that under the equivalence $\text{Fun}^{\text{Ex}}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}, \text{Sp}) \simeq \text{End}^{\text{L}}(\text{Ind}\,\mathcal{C})$, the bimodule $M_n \circ (p^{\text{op}} \times p)$ is equivalently written $\text{Ind}(p)^R \circ M_n \circ \text{Ind}(p)$ where $\text{Ind}(p)^R$ is the right adjoint of $\text{Ind}(p)$ and $p$ is the canonical projection $\text{Lace}(\mathcal{C}, M_1 \oplus ... \oplus M_{n-1}) \to \mathcal{C}$. Hence, by invariance under cyclic permutations, we have

$$\mathrm{P}_1^{\text{fbw}} F^{\text{cyc}}(\text{Lace}(\mathcal{C}, M_1 \oplus ... \oplus M_{n-1}), \widetilde{M_n}) \xrightarrow{\simeq} \mathrm{P}_1^{\text{fbw}} F^{\text{cyc}}(\mathcal{C}, M_n \circ \text{Ind}(p) \circ \text{Ind}(p)^R)$$

Since $F^{\text{cyc}}(\mathcal{C}, -)$ is finitary so is $\mathrm{P}_1^{\text{fbw}} F^{\text{cyc}}(\mathcal{C}, -)$. In particular, it commutes with taking derivatives — i.e. we can apply the so-called *chain rule* — so that we are reduced to take the derivative of $\text{Ind}(p) \circ \text{Ind}(p)^R$ in $M_1, ..., M_i$ and check it is given by:

$$\bigoplus_{f \in \Sigma_{n-1}} M_{f(1)} \circ ... \circ M_{f(n-1)}$$

Note that the case $n = 1$ is Lemma 6.17 where the category being stabilized is $\mathbf{Cat}^{\text{Ex}}_{/\mathcal{C}}$; indeed, we have already seen in Proposition 3.11 that $M \mapsto (p^{\text{op}} \times p)_! \operatorname{map}_{\text{Lace}(\mathcal{C}, M)}$ is equivalent to the composite $\Sigma^\infty \Omega^\infty$ and tracking this formula along the equivalence $\text{Fun}^{\text{Ex}}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}, \text{Sp}) \simeq \text{End}^{\text{L}}(\text{Ind}\,\mathcal{C})$ concludes.

We explain the case $n = 2$ for the sake of sanity (of both the reader and the writer). Denote $p_i : \text{Lace}(\mathcal{C}, M_i) \to \mathcal{C}$ for $i = 1, 2$ and $p_{12} : \text{Lace}(\mathcal{C}, M_1 \oplus M_2) \to \text{Lace}(\mathcal{C}, M_1)$ the canonical projections; since Lace preserves pullback the map $\text{Lace}(\mathcal{C}, M_1 \oplus M_2) \to \mathcal{C}$ factors as $p_1 \circ p_{12}$ hence the expression we are trying to derivative in both variables is given by:

$$\Sigma^\infty \Omega^\infty (M_1 \oplus M_2) \simeq \text{Ind}(p_1) \circ \text{Ind}(p_{12}) \circ \text{Ind}(p_{12})^R \circ \text{Ind}(p_1)^R$$

By Lemma 6.16, we have:

$$\text{Lace}(\mathcal{C}, M_1 \oplus M_2) \simeq \text{Lace}(\text{Lace}(\mathcal{C}, M_1), \text{Ind}(p_1)^R \circ M_2 \circ \text{Ind}(p_1))$$

Hence, $\text{Ind}(p_1)^R \circ M_2 \circ \text{Ind}(p_1)$ is the derivative in the variable $M_2$ of $\text{Ind}(p_{12}) \circ \text{Ind}(p_{12})^R$. Hence, it remains to take the derivative of $\text{Ind}(p_1) \circ \text{Ind}(p_1)^R \circ M_2 \circ \text{Ind}(p_1) \circ \text{Ind}(p_1)^R$ in the variable $M_1$. Another instance of Lemma 6.17 shows that $\text{Ind}(p_1) \circ \text{Ind}(p_1)^R$ has $M_1$ as its derivative; note also that it is not reduced in $M_1$ but and that the constant term is precisely id. In consequence, after linearizing, the expression we pick up is:

$$(M_1 \circ M_2 \circ \text{id}) \oplus (\text{id} \circ M_2 \circ M_1)$$

which is the wanted expression when there are two bimodules.

This affords a computation of the derivatives, but not quite that the square we wanted is cartesian. For this, it would suffice to show that the following square commutes naturally in $M$:

$$\begin{tikzcd}
\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathrm{Lace}(\mathcal{C}, M), \tilde{N}) \arrow[rr, "\simeq"] \arrow[d, "\simeq"] & & \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, N \circ \mathrm{Ind}(p) \circ \mathrm{Ind}(p)^R) \arrow[d] \\
\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, M \oplus N) \arrow[r] & \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, (M \oplus N)^{\circ 2})^{\mathrm{hC}_2} \arrow[r] & \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}(\mathcal{C}, N \circ M)
\end{tikzcd}$$

Note that this is less obvious than it may seem, as most of the those maps do not exist before applying $F^{\mathrm{cyc}}$ or even $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}$, but they arise as the inverses of maps who exist. We will not prove this. □

**Remark 6.19** It follows from Theorem 6.18 that for Verdier-localizing $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$, the first Goodwillie derivative $F^{\mathrm{lace}}$ controls the whole Taylor tower, i.e. if $\eta : F \implies G$ is a natural transformation between such Verdier-localizing functors which induces an equivalence of first Goodwillie derivatives of the laced invariants, then it must also induce an equivalence of the whole Taylor tower.

The functor $\mathrm{K}^{\mathrm{lace}}$ factors through Lace by construction and by Corollary 6.15 its derivative THH is invariant under cyclic permutation (in fact, this is already the case of $\mathrm{K}^{\mathrm{cyc}}$, as we have shown). Hence Theorem 6.18 applies and we have:

**Corollary 6.20** The $n$-homogenous part $\mathrm{D}_n \, \mathrm{K}^{\mathrm{lace}}$ of laced K-theory is $\mathrm{THH}(\mathcal{C}, M^{\otimes n})_{\mathrm{hC}_n}$

This generalizes a result of Lindenstrauss-McCarthy [LM12] when $\mathcal{C} = \mathrm{Perf}(R)$ of a discrete ring $R$, which was generalized by Pancia in his thesis to connective ring spectra [Pan14].

## 6.4 The low-tech polygonic and cyclotomic structures on THH

If $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is Verdier localizing, then, as mentioned in Remark 6.19, the first Goodwillie derivative controls the whole Taylor tower. The goal of this section is to realize this via a different perspective: we will show that by considering the $F(\mathcal{C}, M^{\otimes n})$ as a family, the first Goodwillie derivative acquires extra structure, which we call a genuine polygonic structure, i.e. lifts to a category $\mathrm{GenPgc}(\mathcal{E})$ with a conservative map back to $\prod_n \mathcal{E}$. Moreover, we want to explain why this structure is precisely what is needed to completely recover the whole Taylor tower.

When restricting to bimodules of the form $\mathrm{id}_{\mathcal{C}}$, this structure refines to that of genuine cyclotomic objects of $\mathcal{E}$. To make this precise and rigorous would require even more technology, and since we have already given up on this, we make this section "low-tech" and only sketch the key ingredients that would be required to build the structure.

Our starting point is to remark that there is a *degree $k$ map* as follows:

$$\deg_k \colon \mathrm{Lace}(\mathcal{C}_1 \times ... \times \mathcal{C}_n, (M_1, ..., M_n)) \longrightarrow \mathrm{Lace}((\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\times k}, (M_1, ..., M_n)^{\times k})$$

which is induced by the $k$-fold diagonal. In particular, the right hand side has a $\mathrm{C}_k$-action by permutation which makes the above map into a $\mathrm{C}_k$-equivariant map if the left hand side is given the trivial action. By design, this action is compatible with the trace actions so that for instance

$$\deg_k \colon \mathrm{Lace}(\mathcal{C}^{\times n}, (M, ..., M)) \longrightarrow \mathrm{Lace}(\mathcal{C}^{\times nk}, (M, ..., M)^{\times k})$$

is $\mathrm{C}_n \times \mathrm{C}_k$-equivariant. After applying a Verdier-localizing functor and fiberwise reducing, those products can be turned into tensors (i.e. compositions) as we proved in Lemma 6.10. In particular, the $\mathrm{C}_n$-equivariant maps

$$F^{\mathrm{lace}}(\mathcal{C}^{\times n}, (M, ..., M)) \longrightarrow F^{\mathrm{lace}}(\mathcal{C}^{\times nk}, (M, ..., M)^{\times k})^{\mathrm{hC}_k}$$

induce after taking reduced approximations $\mathrm{C}_n$-equivariant maps:

$$\phi_{n,k}\colon F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n}) \longrightarrow F^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes nk})^{\mathrm{hC}_k}$$

which are such that $\phi_{n,k}(\mathcal{C}, M) = \phi_{1,k}(\mathcal{C}, M^{\otimes n})$ and are natural in the pair $(\mathcal{C}, M)$. Here, $F^{\mathrm{cyc}}$ is the name we have coined in Definition 6.12 for the fiberwise-reducification of $F^{\mathrm{lace}}$.

**Remark 6.21** The structure on $F^{\mathrm{cyc}}$ is a polygonic analogue of the notion of cyclotomic spectra with Frobenius lifts as introduced in [AN20] (see before Lemma 3.7) except we have *a priori* more maps, since $k$ can be not a prime.

Since taking first Goodwillie derivative is functorial, something must happen to those maps after linearizing. Recall that for $X \in \mathrm{Sp}$, the following sequence:

$$(X^{\otimes n})_{\mathrm{h}\Sigma_n} \longrightarrow (X^{\otimes n})^{\mathrm{h}\Sigma_n} \longrightarrow (X^{\otimes n})^{\mathrm{t}\Sigma_n}$$

exhibits its source as the $n$-homogeneous part of its middle term (see 4.29 in [Heu15], which refers to earlier work of [McC01]), and consequently, the right hand side as its $(n-1)$-excisive approximation. When $n = p$ is prime, the right hand side is actually linear by [NS17, Proposition III.1.1] so in fact is also the first derivative.

We claim this latter fact still holds in the laced world:

**Lemma 6.22** Suppose $p$ is prime. Then, the functor $M \mapsto (\mathrm{P}_1^{\mathrm{fbw}}F)(\mathcal{C}, M^{\otimes p})^{\mathrm{tC}_p}$ is exact, and it further is the first Goodwillie derivative of $M \mapsto (\mathrm{P}_1^{\mathrm{fbw}}F)(\mathcal{C}, M^{\otimes p})^{\mathrm{hC}_p}$

**Proof.** There is an exact sequence, natural in the laced category $(\mathcal{C}, M)$:

$$(\mathrm{P}_1^{\mathrm{fbw}}F)(\mathcal{C}, M^{\otimes p})_{\mathrm{hC}_p} \longrightarrow (\mathrm{P}_1^{\mathrm{fbw}}F)(\mathcal{C}, M^{\otimes p})^{\mathrm{hC}_p} \longrightarrow (\mathrm{P}_1^{\mathrm{fbw}}F)(\mathcal{C}, M^{\otimes p})^{\mathrm{tC}_p}$$

We first argue that its target is linear in $M$; this is essentially the same argument as in Proposition III.1.1 of [NS17]: the tensor of bimodule is linear in each variable hence $(\mathrm{P}_1^{\mathrm{fbw}}F)(\mathcal{C}, M_1 \otimes \ldots \otimes M_p)$ sends colimits in each $M_i$ to colimits: if $M_1 = \ldots = M_p$ and the colimit is finite, then one checks that all the terms in the colimit have their $\mathrm{C}_p$-action trivially induced unless they are diagonal. After taking the Tate construction, only the diagonal ones survive hence the result coincide with taking the colimit outside as wanted.

To get the second claim, it suffices to check that the first map of the above exact sequence induces an equivalence on $p$-cross effect, which more or less follows from a similar observation. □

Consequently, on first Goodwillie derivatives, the degree $n$ maps yield $\mathrm{C}_n$-equivariant maps:

$$\psi_{n,p} : (\mathrm{P}_1^{\mathrm{fbw}}F)(\mathcal{C}, M^{\otimes n}) \longrightarrow (\mathrm{P}_1^{\mathrm{fbw}}F)(\mathcal{C}, M^{\otimes np})^{\mathrm{tC}_p}$$

When $M = \mathrm{id}$, remark that $F^{\mathrm{cyc}}(\mathcal{C}, \mathrm{id})$ can be realized as the following geometric realization:

$$\left| \ \ldots \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \ F^{\mathrm{cyc}}(\mathcal{C} \times \mathcal{C}, (\mathrm{id}, \mathrm{id})) \ \substack{\longrightarrow\\[-1ex]\longrightarrow} \ F^{\mathrm{cyc}}(\mathcal{C}, \mathrm{id}) \right|$$

This simplicial object is actually underlying a cyclic object in the sense of Connes, so that this geometric realization acquires a $S^1$-action. Since the maps $F^{\mathrm{cyc}}(\mathcal{C}, \mathrm{id}) \to F^{\mathrm{cyc}}(\mathcal{C}, \mathrm{id}^{\otimes n})$ are compatible with this cyclic realization (i.e. they make the correct diagram commutes and are $\mathrm{C}_n$-equivariant when needed), they induce $S^1$-equivariant $F^{\mathrm{cyc}}(\mathcal{C}, \mathrm{id}) \to F^{\mathrm{cyc}}(\mathcal{C}, \mathrm{id})^{\mathrm{hC}_p}$ so that the previous structure is indeed that of a cyclotomic spectrum when restricted to $(-, \mathrm{id})$.

When $k$ is not a prime $p$, the maps also still exists, but what replaces the Tate construction in this context is the proper Tate construction.

**Definition 6.23** Let $G$ be a group. Then, the *proper Tate construction* is the functor $(-)^{\tau G}$ : $\mathcal{E}^{BG} \to \mathcal{E}$ which is the target of the initial natural transformation with source $(-)^{\mathrm{h}G}$ and target a functor that vanishes on all $G$-objects which are induced from a *proper* subgroup $H$.

Such a universal object exists, see for instance [QS22], and folklore tells that it has some functoriality — we will prove this in a companion paper to [HNS25]. If $G = \mathrm{C}_p$, then there are no proper subgroups save for $\{*\}$ so the proper Tate construction of $\mathrm{C}_p$ is just the usual Tate construction.

**Proposition 6.24** Let $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ be Verdier-localizing, and $n \in \mathbb{N}$, then the functor

$$M \mapsto (\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}})(\mathcal{C}, M^{\otimes n})^{\tau \mathrm{C}_n}$$

is exact, where $(-)^{\tau \mathrm{C}_n}$ denotes the proper Tate construction. In fact, it coincides with the exact approximation of $M \mapsto (\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}})(\mathcal{C}, M^{\otimes n})^{\mathrm{hC}_n}$.

**Proof.** We omit this proof, which is very similar to Lemma 6.22 or the proof of 4.29 in [Heu15], and refer to the fact that it is proven in [HNS25]. □

In consequence, via the universal property of the first derivative, the structure maps of the polygonic object with Frobenius lifts induces $\mathrm{C}_n$-equivariant maps:

$$\psi_{n,k} : (\mathrm{P}_1^{\mathrm{fbw}} F)(\mathcal{C}, M^{\otimes n}) \longrightarrow (\mathrm{P}_1^{\mathrm{fbw}} F)(\mathcal{C}, M^{\otimes nk})^{\tau \mathrm{C}_k}$$

which fit in the following commutative diagram:

$$\begin{tikzcd}
F(\mathcal{C}, M^{\otimes n}) \arrow[r, "\phi_{n,k}"] \arrow[d] & F(\mathcal{C}, M^{\otimes nk})^{\mathrm{hC}_k} \arrow[r] & (\mathrm{P}_1^{\mathrm{fbw}} F)(\mathcal{C}, M^{\otimes nk})^{\mathrm{hC}_k} \arrow[d] \\
(\mathrm{P}_1^{\mathrm{fbw}} F)(\mathcal{C}, M^{\otimes n}) \arrow[rr, dotted, "\psi_{n,k}"] & & (\mathrm{P}_1^{\mathrm{fbw}} F)(\mathcal{C}, M^{\otimes nk})^{\tau \mathrm{C}_k}
\end{tikzcd}$$

We would like to formalizing this structure, and it is reasonable to expect that it would come in fact from a genuine version of *polygonic spectra*. Here, we use genuine in a corrupted sense, meaning that eventually, the structure we introduce will be equivalent to genuine construction which have already been proved but we will personally never work in the genuine equivariant world, especially in the context of this work which will not touch this comparison.

Let us first recall the non-genuine definitions of [KMN23]; it will be convenient to have the same flavors as in *loc. cit.* so we introduce as well the following definition:

**Definition 6.25** A truncation set $T$ is a subset of $\mathbb{N}_{\geq 1}$ such that $xy \in T$ implies $x \in T$ and $y \in T$.

**Definition 6.26 — Definition 2.6 of [KMN23].** Let $\mathcal{C}$ be a presentable stable category. The category $\mathrm{Pgc}_T(\mathcal{C})$ of $T$-polygonic objects in $\mathcal{C}$ is given by the following lax-equalizer

$$\mathrm{LEq}\left(\prod_{n \in T} \mathcal{C}^{B\mathrm{C}_n} \rightrightarrows \prod_p \prod_{k \in T/p} \mathcal{C}^{B\mathrm{C}_k}\right)$$

where the top map is the identity on each component and the second map takes the $p$-Tate construction on each component. If $T = \mathbb{N}_{\geq 1}$, we write simply $\mathrm{Pgc}(\mathcal{C})$.

In particular, objects of $\mathrm{Pgc}_T(\mathcal{C})$ can be described as a collection $X_n \in \mathcal{C}^{B\mathrm{C}_n}$ for each $n \in T$ with maps $\phi_{p,n} : X_n \longrightarrow X_{np}^{\mathrm{tC}_p}$ whenever $np \in T$.

General results on lax-equalizers (see [NS17, Proposition II.1.5]) imply the following:

**Proposition 6.27** The category $\mathrm{Pgc}_T(\mathcal{C})$ is presentable stable, and for $n \in T$, the functors $\mathrm{Pgc}_T(\mathcal{C}) \to \mathcal{C}^{B\mathrm{C}_n}$ are colimit-preserving.

There is a functor $\mathrm{triv} : \mathcal{C} \to \mathrm{Pgc}_T(\mathcal{C})$ which can be informally described as sending $X$ to the tuple of $(X)$ where $X$ is viewed with the trivial $\mathrm{C}_n$-action in $\mathcal{C}^{B\mathrm{C}_n}$, $n \in T$, and equipped with the canonical maps $X \to X^{\mathrm{hC}_p} \to X^{\mathrm{tC}_p}$ that exists because the action on $X$ is trivial.

**Definition 6.28** We write $\mathrm{TR}_T$ for the functor $\mathrm{Pgc}_T(\mathrm{Sp}) \to \mathrm{Sp}$ given by $\mathrm{map}(\mathbb{S}^{\mathrm{triv}}, -)$. This is the right adjoint to triv. If $T = \mathbb{N}_{\geq 1}$, then we write simply TR.

There is a functor $i : \mathrm{Cyc}(\mathrm{Sp}) \to \mathrm{Pgc}(\mathrm{Sp})$, where on the left hand side we denote the category of cyclotomic spectra as in [NS17], which can be informally described forgetting from the $S^1$-action down to $\mathrm{C}_n$-actions everywhere — the details are spelled out in Construction 2.16 of [KMN23]. The following is proven in the same section of *loc. cit.*:

**Lemma 6.29** The functor $i : \mathrm{Cyc(Sp)} \to \mathrm{Pgc(Sp)}$ admits a right adjoint, denoted $R : \mathrm{Pgc(Sp)} \to \mathrm{Cyc(Sp)}$. Moreover, since triv factors through $i$, we have an equivalence $\mathrm{TR} \simeq \mathrm{TC} \circ R$.

In the previous sections, we have worked enough to get the following:

**Theorem 6.30** Let $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be Verdier-localizing, then the functor $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ admits a canonical lift to $\mathrm{Pgc}(\mathcal{E})$ which refines to a cyclotomic structure when restricted to objects of the form $(\mathcal{C}, \mathrm{id})$. Moreover, the canonical map

$$F^{\mathrm{cyc}} \longrightarrow \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{cyc}}$$

is a map of polygonic spectra when the left hand side is endowed with its polygonic structure with Frobenius lifts.

In particular, there is a well-defined functor $\mathrm{THH} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{PgcSp}$ which recovers the usual cyclotomic structure on $\mathrm{THH}(\mathcal{C})$ for $\mathcal{C}$ a stable category, as well as functors $\mathrm{TC} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ and $\mathrm{TR} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ receiving natural transformations from respectively K and $\mathrm{K}^{\mathrm{lace}}$.

**Proof.** We have seen in Theorem 6.18 how to provide a cyclic action on $(\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})(\mathcal{C}, M^{\otimes n})$ (see also the remark below); we also have provided the structure maps

$$\psi_{n,p} : (\mathrm{P}_1^{\mathrm{fbw}} F)(\mathcal{C}, M^{\otimes n}) \longrightarrow (\mathrm{P}_1^{\mathrm{fbw}} F)(\mathcal{C}, M^{\otimes np})^{\mathrm{tC}_p}$$

hence the first claim. When $M = \mathrm{id}$, the cyclic actions come from a common $S^1$-action because the Bar construction that defines cyc (see Proposition 5.20 and remark that every trace-like invariant $F$ satisfies $F \simeq \mathrm{cyc}(F)$) is actually underlying a cyclic object, and since the maps are compatible with this geometric realization, they also become $S^1$-equivariant.

Finally, the trivial polygonic structure was built with maps to the Tate factoring through homotopy fixed points; hence the natural transformation $F^{\mathrm{lace}} \to (\mathrm{P}_1^{\mathrm{fbw}} F)$ makes the wanted squares commute. □

**Remark 6.31** Starting from Proposition 6.24, we have made the choice of trying to understand what structure survives the linearization process. In an other direction, note that

$$(\mathrm{P}_1^{\mathrm{fbw}} F)(\mathcal{C}, M^{\otimes n})^{\mathrm{hC}_n}$$

is $n$-excisive so that the higher excisive approximations (or more precisely, the actual Ind-object $\mathrm{P}_\bullet^{\mathrm{fbw}} F$) acquires more structure. This induces the structure of a polygonic spectrum with Frobenius lifts on the limit $\mathrm{P}_\infty^{\mathrm{fbw}} F$.

Unfortunately, the polygonic structure on the derivative is not enough information to recover the whole Taylor tower in general, though it gets rather close. If the underlying spectra of $\mathrm{P}_1^{\mathrm{fbw}}$ are all bounded-below, then we will show in the companion paper to [HNS25] that this is enough to recover the limit of the Taylor tower by taking TR and more generally, $\mathrm{P}_n^{\mathrm{fbw}} F^{\mathrm{lace}}$ is given by $\mathrm{TR}_{[n]}$, where $[n]$ is the truncation set $\{1, ..., n\}$.

There is however a refinement of this structure which carries all of the information, and that we call the genuine polygonic structure. The genuine version $\mathrm{Pgc}_T^{gen}(\mathcal{C})$ of polygonic objects is a version of this construction which refines the maps to have target as the *proper* Tate constructions, as well as having genuine actions on every $X_i$. Unfortunately, already with the proper Tate construction, this is harder to build because we also want to have compatibility data between the proper Tate construction for say $n \in T$ and $k \mid n$, hence a lax-equalizer does not suffice.

We have made the choice in this section to avoid such compatibility issues. Hence, we will not define this genuine refinement here. We will state formally all of the relevant statements in the next section.

# 7 The structure of THH of the bicategory $\mathbf{Cat}^{\mathrm{Ex}}$

This section is the factual counterpart of the previous section. Here we collect all the main results proven in [HNS25], although we systematically omit proofs.

To prove the following results, one key idea is to categorify the problem: we will build a category out of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ that implements the wanted action on $(\mathcal{C}, M^{\otimes n})$ (or more precisely a lax-version thereof) and a further category where the polygonic maps exist. To construct this category, we imitate the construction of THH in a categorified world. This resembles, but does not quite coincide with the theory of shadows of [PS13] even in its higher-categorical version of [HR23]; instead we follow ideas proposed by Thomas Nikolaus in [Nik18] and continued in [Ram24b, Ram24a]. We also note that in *loc. cit.*, Maxime Ramzi has announced that he could compare the two theories (in a suitable sense), so we are not worried about the differences.

## 7.1 Simplicial, cyclic and epicyclic tangent bundles

We built $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ as the unstraightening of the functor sending $\mathcal{C}$ to its category of $\mathcal{C}$-bimodules. In a situation where we want to deal with more than one bimodule, the following is a natural generalization:

**Definition 7.1** Let $n \in \mathbb{N}$, then we define $\Lambda_n\mathbf{Cat}^{\mathrm{Ex}}$ to be the category obtained by unstraightening the contravariant functor given by:

$$\begin{aligned} (\mathbf{Cat}^{\mathrm{Ex}})^{n+1} &\longrightarrow \mathbf{CAT} \\ (\mathcal{C}_0, ..., \mathcal{C}_n) &\longmapsto \prod_{i \in \mathbb{Z}/(n+1)\mathbb{Z}} \mathrm{Fun}^{\mathrm{Ex}}({\mathcal{C}_{i+1}}^{\mathrm{op}} \otimes \mathcal{C}_i, \mathrm{Sp}) \end{aligned}$$

with functoriality induced by restriction. It comes with a cartesian fibration $\Lambda_n\mathbf{Cat}^{\mathrm{Ex}} \to (\mathbf{Cat}^{\mathrm{Ex}})^{n+1}$ which happens to also be cocartesian, and cocartesian transitions functors induced by left Kan extension.

For $n = 0$, we thus recover precisely $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. Generally, an object of $\Lambda_n\mathbf{Cat}^{\mathrm{Ex}}$ can be thought as a marked cyclic graph with $n+1$ vertices $(\mathcal{C}_0, ..., \mathcal{C}_n)$, the marking corresponding to choosing who is the zeroth vertex, and arrows given by bimodules $M_i : \mathcal{C}_i \to \mathrm{Ind}(\mathcal{C}_{i+1})$. Arrows of $\Lambda_n\mathbf{Cat}^{\mathrm{Ex}}$ are given by functors $\mathcal{C}_i \to \mathcal{D}_i$ and some lax-commutative squares similar to the situation of Remark 3.17.

There is extra structure relating the different $\Lambda_n\mathbf{Cat}^{\mathrm{Ex}}$, namely composing bimodules or adding Yoneda embeddings within graphs, which yields the following extra structure:

**Lemma 7.2** The construction $[n] \mapsto \Lambda_n\mathbf{Cat}^{\mathrm{Ex}}$ refines to a simplicial category. We denote $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ its cocartesian unstraightening.

The category $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ has objects cyclic graphs of $n$ elements, for arbitrary $n \in \mathbb{N}^*$. We remark that unlike $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ or even $\Lambda_n\mathbf{Cat}^{\mathrm{Ex}}$, the fibration $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}) \to \Delta^{\mathrm{op}}$ is not cartesian (though there are some cartesian transition functors for some special maps of $\Delta$).

**Remark 7.3** Understanding unstraightening as a lax-colimit, $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ is thus the lax geometric realization of some 2-categorical version of the cyclic Bar construction. In particular, one should understand $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ as a 2-categorical version of the bare spectrum $\mathrm{THH}(\mathcal{C}, M)$. This is the category denoted $\Lambda^{st}_{\Delta}$ in [Ram24b].

We let $\Lambda$ denote *Connes' cyclic category*, see appendices of [NS17, Ram24b] for a modern treatment (as well as some recollections in [HNS25]). There is a faithful functor $\delta : \Delta \to \Lambda$ which is a bijection on objects (note however that there is a shift in notations and $[n]_\Lambda$ is to be thought as a graph with $n + 1$ vertices), and such that every morphism $f \in \Lambda$ is of the form $\alpha \circ \delta(g)$ with $g$ a map in $\Delta$ and $\alpha$ an equivalence, corresponding to a rotation of the graph.

**Proposition 7.4** The simplicial $\Delta^{\mathrm{op}} \to \mathbf{Cat}$ mapping $[n] \mapsto \Lambda_n\mathbf{Cat}^{\mathrm{Ex}}$ upgrades to a cyclic object, i.e. a functor $\Lambda^{\mathrm{op}} \to \mathbf{Cat}$. We denote $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$ the cocartesian unstraightening of the latter.

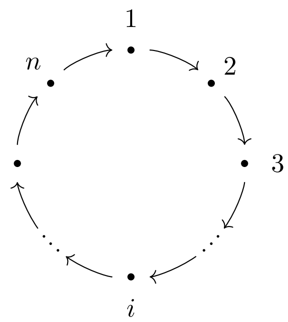
There is a functor $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$ which is a map of cocartesian fibrations; in fact, the former is pulled from the latter along $\delta^{\mathrm{op}}$. The category $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$ allows for extra freedom: cocartesian transition functors over rotations allow one to rotate cyclic graphs with $n$

elements; but since rotations are invertible, this has identified some graphs together. In particular, in $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$, graphs with $n$ elements whose every vertex is equivalent to the same category $\mathcal{C}$ and whose elementary edges are the same bimodule $M$ carry a $\mathrm{C}_n$-action. Thus, $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$ is a 2-categorical version of THH where we have implemented the cyclic actions on objects of the form $(\mathcal{C}, M^{\otimes n})$

If $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$ was truly a THH-kind of construction, then we would expect extra structure in the form of lax-Tate diagonal, resembling the $\mathrm{C}_n$-equivariant maps

$$\mathrm{THH}(\mathcal{C}, M^n) \to \mathrm{THH}(\mathcal{C}, M^{np})^{\mathrm{tC}_p}$$

To do this, we consider a refinement of $\Lambda$ first introduced by Goodwillie: the epicyclic category $\Lambda^{\mathrm{epi}}$ (see also [McC23] though we note that critically, the claim of Example 2.1.11 that the degree is a cartesian fibration fails, the corrected claim is and subsequent theory is developed and proved in [HNS25]) By definition, $\Lambda^{\mathrm{epi}}$ is the full subcategory of $\mathbf{Cat}$ spanned by the cyclic graph categories $\mathbb{T}_n$ which are pictorially represented by:




In particular, maps in $\Lambda^{\mathrm{epi}}$ induce $k$-fold covers of the circle upon realization, for some $k \geq 1$. We will say that such a $k$ is the degree of the map in $\Lambda^{\mathrm{epi}}$, and taking degrees upgrades to a functor $\deg : \Lambda^{\mathrm{epi}} \to B\mathbb{N}^{\times}$, i.e. is multiplicative.

There is a functor $\Lambda \to \Lambda^{\mathrm{epi}}$ which sends $[n]_{\Lambda}$ to $\mathbb{T}_n$; this functor is faithful and a bijection on objects. Maps in the image are precisely the degree 1 maps, i.e. $\Lambda$ is the fiber of the degree functor.

**Proposition 7.5** The cyclic functor $[n] \mapsto \Lambda_n \mathbf{Cat}^{\mathrm{Ex}}$ upgrades to an epicyclic object, i.e. a functor $(\Lambda^{\mathrm{epi}})^{\mathrm{op}} \to \mathbf{Cat}$. We denote $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}})$ the cocartesian unstraightening of the latter.

Again, the above implies that the cocartesian fibration $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}}) \to \Lambda^{\mathrm{op}}$ is pulled back from the fibration $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}}) \to (\Lambda^{\mathrm{epi}})^{\mathrm{op}}$. In particular, we have a chain of functors

$$\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \longrightarrow \mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}) \longrightarrow \mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}}) \longrightarrow \mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}})$$

Each of those categories is enriched in $\mathbf{Cat}^{\mathrm{Ex}}$ and so considering the internal functor category from the image of $(\mathrm{Sp}^{\mathrm{fin}}, \mathrm{id})$, we define variations of Lace, namely $\mathrm{Lace}^{\Delta}$, $\mathrm{Lace}^{\Lambda}$ and $\mathrm{Lace}^{\mathrm{epi}}$. Those laces are compatible with the restriction: in particular, any laced-invariant can be canonically promoted to a functor $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathcal{E}$.

**Definition 7.6** A functor $\mathrm{THH}^{*}(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathcal{E}$ with $* \in \{\Delta, \Lambda\}$ is said to be *cyclic invariant* or *invariant under cyclic permutations* if it inverts cocartesian edges. The full subcategory of such functors will be denoted $\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^{*}(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E})$.

Note that we also have a definition of cyclic-invariance in $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ in Definition 6.9, we write similarly $\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$. Our goal is now to compare the differences between the notions of trace-invariance we have defined. Our first result in this direction in [HNS25] is as follows:

**Proposition 7.7** The functor $\nu_{\Delta} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ induces an equivalence for every $\mathcal{E}$:

$$\nu_{\Delta}^{*} : \mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{cyc}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})$$

In particular, $\mathrm{K}^{\mathrm{cyc}}$ and THH canonically lift to the left hand side category.

**Remark 7.8** It also holds that $\nu_\Delta : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})$ identifies the category of trace-like functors $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ with the subcategory of functors $\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathcal{E}$ which invert cocartesian edges over surjections of $\Delta$.

Note that the category $\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E})$ has more structure:

**Lemma 7.9** The category $\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E})$ carries a canonical action of $S^1$.

**Proof.** Recall that $\mathrm{Un}(F)[\mathrm{coCart}^{-1}]$, the localization of the unstraightening of a functor $F$ at its cocartesian edges, computes the colimit of a functor $F : \mathcal{I} \to \mathbf{Cat}$ (whereas $\mathrm{Un}(F)$ is a lax version thereof). In particular, $\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})[\mathrm{coCart}^{-1}]$ carries a canonical $S^1$-action as the geometric realization of the underlying simplicial object of a cyclic object. Thus, via the equivalence

$$\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E}) \simeq \mathrm{Fun}(\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})[\mathrm{coCart}^{-1}], \mathcal{E})$$

we get a canonical $S^1$-action on the left hand side by precomposition. □

In fact, for a functor $\Lambda^{\mathrm{op}} \to \mathcal{E}$, one can show (and we do in [HNS25]) that the colimit of the cyclic object is given by taking first the colimit of the underlying simplicial object, and then the colimit of the induced $S^1$-action, i.e. homotopy orbits for this action. The following is now completely formal:

**Proposition 7.10** The functor $\nu_\Lambda : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{THH}^\Lambda(\mathbf{Cat}^{\mathrm{Ex}})$ induces an equivalence for every $\mathcal{E}$:

$$\nu_\Lambda^* : \mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^\Lambda(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{cyc}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})^{\mathrm{h}S^1}$$

Hence, fixed points under the $S^1$-action precisely corresponds to those invariants where the two equivalences $F(\mathcal{C}, M \circ N) \simeq F(\mathcal{D}, N \circ M)$ are inverse to one another, and similar higher conditions.

**Remark 7.11** It holds that for a Verdier-localizing invariant $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$, the fiberwise-reduction $F^{\mathrm{cyc}}$ of its lacing is automatically already a $S^1$-fixed point under the above action.

In essence, this is captured by the first part of Lemma 6.10: the functor Lace canonically identifies the two size-2 graph categories but this can also be seen by a phenomenon we already mentioned: Lace lifts to $\mathrm{THH}^\Lambda(\mathbf{Cat}^{\mathrm{Ex}})$. In particular, $F \circ \mathrm{Lace}$ admits a canonical extension to $\mathrm{THH}^\Lambda$ such that, as we will see later, trace-invariance is fully-captured by the restriction on $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, so that forcing trace-invariance commutes with restriction.

In both $\mathrm{THH}^\Delta(\mathbf{Cat}^{\mathrm{Ex}})$ and $\mathrm{THH}^\Lambda(\mathbf{Cat}^{\mathrm{Ex}})$, there is a fiberwise-linearization operation and they are compatible with restrictions (i.e. are performed fiberwise). Now, given a Verdier-localizing invariant $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}^{BS^1}$, one can build a functor

$$F^{\Lambda-lace} : \mathrm{THH}^\Lambda(\mathbf{Cat}^{\mathrm{Ex}}) \longrightarrow \Lambda^{op} \times \mathbf{Cat}^{\mathrm{Ex}} \longrightarrow BS^1 \times \mathbf{Cat}^{\mathrm{Ex}} \xrightarrow{F} \mathcal{E}$$

using that $|\Lambda^{\mathrm{op}}| \simeq BS^1$ and the $\mathrm{Lace}^\Lambda$ already mentioned, for instance in Remark 7.11. The linearization of $F^{\Lambda-lace}$ is automatically a trace-invariant functor $\mathrm{THH}^\Lambda(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathcal{E}$ and thus restricts to a fixed point under the $S^1$-action on trace-invariant functors out of $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$. Since the linearization is done fiberwise, this is also ultimately the structure of a $S^1$-fixed point on the linearization $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$.

**Theorem 7.12** If $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}^{BS^1}$ is Verdier-localizing, then the functor $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}$ canonically promotes to a functor

$$\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}} : \mathrm{THH}^\Lambda(\mathbf{Cat}^{\mathrm{Ex}}) \longrightarrow \mathcal{E}$$

so that in particular, $(\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})(\mathcal{C}, \mathrm{id})$ carries a $S^1$-action. Moreover, the canonical map

$$F(\mathcal{C}) \longrightarrow (\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})(\mathcal{C}, \mathrm{id})$$

refines to a $S^1$-equivariant map.

Applied to K with the trivial $S^1$-action, one gets that THH naturally promotes to a functor on $\mathrm{THH}^{\Lambda}(\mathbf{Cat}^{\mathrm{Ex}})$ and that the usual Dennis trace map is $S^1$-equivariant.

## 7.2 Polygonic and cyclotomic structure on the first derivative

We now turn to the epicyclic $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}})$; our first warning is that the story cannot be the same: we do not want that the cocartesian lifts of the degree $k$ maps get inverted since they are supposed to model the maps in the polygonic structure, which are not equivalences. Instead, what we want to enforce is some equivariance, as in the functor $\deg_k$ of the introduction of 6.4.

First, let us give an overview of the category of genuine polygonic objects. We write $\mathrm{Orb}_{\mathrm{fin}}(\mathbb{Z})$ for the category of finite $\mathbb{Z}$-orbits whose underlying groupoid is the disjoint union of the $B\mathrm{C}_n$. There is a functor $\mathrm{Orb}_{\mathrm{fin}}(\mathbb{Z}) \to \Lambda^{\mathrm{epi}}$ which is faithful, a bijection on objects and a map of $\Lambda^{\mathrm{epi}}$ is in the image if and only if it is Kan fibration. We let

$$\mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E}) := \mathrm{Fun}(\mathrm{Orb}_{\mathrm{fin}}(\mathbb{Z})^{\mathrm{op}}, \mathcal{E})$$

be the category of *polygonic objects in $\mathcal{E}$ with Frobenius lifts.* An object $X \in \mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})$ contains in particular the datum of a $\mathrm{C}_n$-object $X(n)$ for every $n \geq 1$ as well as $\mathrm{C}_{nk}$-equivariant maps $X(n) \longrightarrow X(nk)$ where the $\mathrm{C}_k \simeq \mathrm{C}_{nk}/\mathrm{C}_n$ action on the left is trivial, so that in particular, it gives rises to $\mathrm{C}_n$-equivariant maps

$$X(n) \longrightarrow X(nk)^{\mathrm{hC}_k}$$

which we call the $k^{th}$ Frobenius lift of $X(n)$.

**Definition 7.13** A functor $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathcal{E}$ is said to be *cyclic-invariant* if it inverts cocartesian edges over degree one maps of $\Lambda^{\mathrm{epi}}$.

Let us introduce the Witt *monoid* $\mathbb{W} := S^1 \rtimes \mathbb{N}^{\times}$ where the action is by rotation; there is a localization $\Lambda^{\mathrm{epi}} \to B\mathbb{W}$ where we invert precisely the degree 1 maps. The following will be shown in [HNS25]:

**Lemma 7.14** The category $\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E})$ carries a canonical oplax $\mathbb{W}^{\mathrm{op}}$-action.

**Sketch of proof.** The idea is roughly the same as in 7.9, but it is marred by the fact that $\mathbb{W}$ is not a group, hence we have to be careful and work with lax limits/colimits.

Remark that $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}})[\deg_1^{-1}]$ is fibered over $B\mathbb{W}^{\mathrm{op}}$ and that this fibration stays cocartesian. It thus holds that $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}})[\deg_1^{-1}]$ is the lax colimit of the functor it classifies, which is precisely a $\mathbb{W}^{\mathrm{op}}$-action on $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})[\mathrm{coCart}^{-1}]$. □

We will not define the words "oplax action" but let us still say the following: there is a 2-category $F_{oplax}(B\mathbb{W}^{\mathrm{op}})$ which realizes the free oplax construction on $B\mathbb{W}^{\mathrm{op}}$, which is dual to the free lax construction (usually realized by the enveloping algebra construction). An oplax $\mathbb{W}^{\mathrm{op}}$-action on some category $\mathcal{C}$ is a (strong) 2-functor

$$F_{oplax}(B\mathbb{W}^{\mathrm{op}}) \longrightarrow \mathbf{Cat}$$

with value at the point given by $\mathcal{C}$. Note that in general, $B\mathbb{W}^{\mathrm{op}} \to F_{oplax}(B\mathbb{W}^{\mathrm{op}})$ is only an oplax 2-functor so that an oplax $\mathbb{W}^{\mathrm{op}}$-action need not yield an honest $\mathbb{W}^{\mathrm{op}}$-action.

**Proposition 7.15** The functor $\nu_{epi} : \mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}) \to \mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}})$ induces a functor

$$\nu_{epi}^{*} : \mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E}) \xrightarrow{\ \simeq\ } \mathrm{Lax}_{\mathrm{h}\mathbb{W}}(\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}), \mathcal{E}))$$

where the right hand side denotes the *lax limit* of the 2-functor $F_{oplax}(B\mathbb{W}^{\mathrm{op}}) \to \mathbf{Cat}$ encoding the oplax $\mathbb{W}^{\mathrm{op}}$-action of Lemma 7.14. Moreover, we have a conservative:

$$\mathrm{Lax}_{\mathrm{h}\mathbb{W}^{\mathrm{op}}}(\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}}),\mathcal{E})) \longrightarrow \mathrm{Fun}^{\mathrm{cyc}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}},\mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E}))$$

In particular, $\mathrm{K}^{\mathrm{cyc}}$ lifts to a functor on $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{GenPgc}^{\mathrm{Fr}}(\mathrm{Sp})$.

The second map of the above theorem is built in a similar manner to the discussion for $F^{\Lambda-lace}$ before Theorem 7.12.

It is critical that there is no way in general to fiberwise-linearize in $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}})$ *while* preserving the cyclic-invariance condition (precisely because it involves highly non-exact operations). However, since we have produced a functor $F^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})$, we can try to understand what happens to the extra-structure.

**Proposition 7.16** Let $\mathcal{E}$ be a presentable stable category. There is a category $\mathrm{GenPgc}(\mathcal{E})$ of *genuine polygonic objects* fitting in a triangle

$$\begin{tikzcd} \mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E}) \arrow[rr] \arrow[dr] & & \mathrm{GenPgc}(\mathcal{E}) \arrow[dl] \\ & \prod_n \mathcal{E}^{BC_n} & \end{tikzcd}$$

where the diagonal maps are colimit-preserving and conservative and the horizontal functor is exact. Moreover, there is a conservative functor

$$\mathrm{GenPgc}(\mathcal{E}) \longrightarrow \mathrm{Eq}\left(\prod_n \mathcal{E}^{BC_n} \underset{(-)^{\tau C_k}}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_k \prod_n \mathcal{E}^{BC_n}\right)$$

whose target can informally be described as the category of sequences $X_n \in \mathcal{E}^{BC_n}$ with maps $X_n \to X_{nk}^{\tau C_k}$ but no higher coherences between the maps.

In particular, when $\mathcal{E} = \mathrm{Sp}$, there is a functor $\mathrm{GenPgc}(\mathrm{Sp}) \to \mathrm{PgcSp}$ with target the category of polygonic spectra of [KMN23] which restricts to an equivalence on bounded-below objects.

The construction of $\mathrm{GenPgc}(\mathcal{E})$ can be understood from the perspective of oplax action: it is the lax limit of some oplax $\mathbb{N}^{\mathrm{op}}$-action on $\mathcal{E}$ encoding the relations between the proper Tate constructions $(-)^{\tau C_n}$ for varying $n$. This is what makes this approach quite powerful, but we have to pay a steep cost: develop a theory sturdy enough to talk about oplax actions and their lax limits.

There is a colimit-preserving functor $\mathrm{triv} : \mathcal{E} \to \mathrm{GenPgc}(\mathcal{E})$ which endows an object with the *trivial* polygonic structure, obtained by composing the trivial polygonic structure with Frobenius lifts with the top map of the triangle of the previous theorem.

**Definition 7.17** We let $\mathrm{TR} : \mathrm{GenPgc}(\mathcal{E}) \to \mathcal{E}$ denote the right adjoint to triv.

Specializing to Sp, the above theorem guarantees that TR recovers the usual TR of [KMN23] for uniformly bounded-below spectra with a polygonic structure.

The key insight is that when linearizing a functor $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ which lifts to $\mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})$, the linearization might not land in $\mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})$ but it will stay in the larger $\mathrm{GenPgc}(\mathcal{E})$. Precisely, this is because even if $F$ is fiberwise-exact, $F(\mathcal{C}, M^{\otimes n})^{\mathrm{h}C_n}$ will *not* be linear but at best $n$-excisive. However, $F(\mathcal{C}, M^{\otimes n})^{\tau C_n}$ will be so that the whole of the further map to the proper Tate construction will survive linearization.

**Theorem 7.18** Let $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a Verdier-localizing functor to a stable $\mathcal{E}$. Then, there is a canonical, functorial lift of $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ to

$$\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{GenPgc}(\mathcal{E})$$

Moreover, the natural transformation $F^{\mathrm{lace}} \to \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}$ lifts to a map of functors $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to$

GenPgc($\mathcal{E}$) where $F^{\mathrm{lace}}$ is given the trivial genuine polygonic structure (which is in particular, a genuine polygonic structure with Frobenius lifts). In consequence, there is a natural transformation

$$F^{\mathrm{lace}}(\mathcal{C}, M) \longrightarrow \mathrm{TR}((\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})(\mathcal{C}, M))$$

of functors $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$, which we call the polygonic trace map.

In particular, as a consequence of this result, we get a functor $\mathrm{THH} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{GenPgc}(\mathrm{Sp})$ valued in genuine polygonic spectra and a factorization of the trace map from laced K-theory to THH as follows:

$$\mathrm{K}^{\mathrm{lace}}(\mathcal{C}, M) \longrightarrow \mathrm{TR}(\mathrm{THH}(\mathcal{C}, M)) \longrightarrow \mathrm{THH}(\mathcal{C}, M)$$

When restricted to bounded-below polygonic spectrum, this structure on $\mathrm{THH}(\mathcal{C}, M))$ recovers the one built in [KMN23], but in general our factorization is finer.

**Remark 7.19** Note that nowhere do we claim that THH or more generally the first derivative of a functor of the form $F^{\mathrm{lace}}$ lifts to $\mathrm{THH}^{\mathrm{epi}}(\mathbf{Cat}^{\mathrm{Ex}})$. Indeed, given that their polygonic structure does not have Frobenius lifts, this would be contradictory with 7.15; this is inline with a previous comment saying that one cannot linearize cyclic-invariants on the epicyclic bundle.

Let us conclude this section by working out a similar phenomenon as in Proposition 7.10: we have seen that $S^1$-fixed points cyclic-invariants of $\mathrm{THH}^{\Delta}(\mathbf{Cat}^{\mathrm{Ex}})$ acquire a $S^1$-action on the diagonal $F(\mathcal{C}, \mathrm{id})$. We now investigate the extra-structure granted by the datum of being a lax $\mathbb{W}^{\mathrm{op}}$-fixed point.

**Definition 7.20** We write $\mathrm{GenCyc}^{\mathrm{Fr}}(\mathcal{E})$ for the functor category $\mathrm{Fun}(B\mathbb{W}^{\mathrm{op}}, \mathcal{E})$, and we call it the category of *genuine cyclotomic spectra with Frobenius lifts.*

Remark that there is a functor $\mathrm{GenCyc}^{\mathrm{Fr}}(\mathcal{E}) \to \mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})$ by precomposing by the functor $\mathrm{Orb}_{\mathrm{fin}}(\mathbb{Z})^{\mathrm{op}} \to (\Lambda^{\mathrm{epi}})^{\mathrm{op}} \to B\mathbb{W}^{\mathrm{op}}$. Moreover, $\mathrm{GenCyc}^{\mathrm{Fr}}(\mathcal{E})$ also canonically maps to $\mathcal{E}^{BS^1}$.

**Lemma 7.21** There is a functor

$$\mathrm{Lax}_{\mathrm{h}\mathbb{W}^{\mathrm{op}}}(\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})) \longrightarrow \mathrm{Fun}(\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{GenCyc}^{\mathrm{Fr}}(\mathcal{E}))$$

which is such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{Fun}^{\mathrm{cyc}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})) \arrow[dr, "L^*"] & \\
\mathrm{Lax}_{\mathrm{h}\mathbb{W}^{\mathrm{op}}}(\mathrm{Fun}^{\mathrm{cyc}}(\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}, \mathcal{E})) \arrow[ur] \arrow[dr] & & \mathrm{Fun}(\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})) \\
 & \mathrm{Fun}(\mathbf{Cat}^{\mathrm{Ex}}, \mathrm{GenCyc}^{\mathrm{Fr}}(\mathcal{E})) \arrow[ur] &
\end{tikzcd}$$

where the bottom right arrow postcomposes by the functor $\mathrm{GenCyc}^{\mathrm{Fr}}(\mathcal{E}) \to \mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E})$ and $L$ denotes as usual the cotangent complex.

In particular, when evaluated at a laced category of the form $(\mathcal{C}, \mathrm{id})$, cyclic K-theory $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, \mathrm{id})$ enjoys the structure of a genuine cyclotomic spectrum with Frobenius lifts. Let us now describe what happens to this structure, and more generally the structure of $F^{\mathrm{cyc}}(\mathcal{C}, \mathrm{id})$ when linearizing. It will be captured by the category introduced in the following proposition:

**Proposition 7.22** Let $\mathcal{E}$ be a presentable stable category. There is a category $\mathrm{GenCyc}(\mathcal{E})$ of *genuine cyclotomic objects* factorizing the canonical map

$$\mathrm{GenCyc}^{\mathrm{Fr}}(\mathcal{E}) \longrightarrow \mathrm{GenCyc}(\mathcal{E}) \longrightarrow \mathcal{E}^{BS^1}$$

such that the second functor is colimit-preserving and conservative. Moreover, there is a conservative functor

$$\mathrm{GenCyc}(\mathcal{E}) \longrightarrow \mathrm{Eq}\left(\mathcal{E}^{BS^1} \underset{(-)^{\tau C_k}}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_k \mathcal{E}^{BS^1}\right)$$

whose target can informally be described as the category whose object is $X \in \mathcal{E}^{BS^1}$ with maps $X \to X^{\tau C_k}$ but no higher coherences between the maps.

Moreover, there is a left adjoint functor $\mathrm{GenCyc}(\mathcal{E}) \to \mathrm{GenPgc}(\mathcal{E})$ which fits in a commutative square

$$\begin{tikzcd} \mathrm{GenCyc}^{\mathrm{Fr}}(\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{GenCyc}(\mathcal{E}) \arrow[d] \\ \mathrm{GenPgc}^{\mathrm{Fr}}(\mathcal{E}) \arrow[r] & \mathrm{GenPgc}(\mathcal{E}) \end{tikzcd}$$

**Remark 7.23** When $\mathcal{E} = \mathrm{Sp}$, $\mathrm{GenCyc}(\mathrm{Sp})$ recovers the usual category of genuine cyclotomic spectra, as in [NS17]. In particular, on bounded below spectra, it also recovers the low-coherence model of *loc. cit.*

There is a left adjoint functor $\mathrm{triv} : \mathcal{E} \to \mathrm{GenCyc}(\mathcal{E})$ which endows an object with the *trivial* cyclotomic structure.

**Definition 7.24** We let $\mathrm{TC} : \mathrm{GenCyc}(\mathcal{E}) \to \mathcal{E}$ denote the right adjoint to triv. For bounded-below spectra with a cyclotomic structure, this recovers the usual TC of [NS17].

We note that if $R$ is the right adjoint to $\mathrm{GenCyc}(\mathcal{E}) \to \mathrm{GenPgc}(\mathcal{E})$, then there is an equivalence

$$\mathrm{TC} \circ R \simeq \mathrm{TR}$$

The final result of this section tells of the interplay between the cyclotomic structure and the polygonic structure:

**Theorem 7.25** Let $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a Verdier-localizing functor to a stable $\mathcal{E}$. Then, there exists a dotted arrow such that lift $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{GenPgc}(\mathcal{E})$ of Theorem 7.18 fits in a commutative square

$$\begin{tikzcd} \mathbf{Cat}^{\mathrm{Ex}} \arrow[r] \arrow[d, "\mathcal{C} \mapsto (\mathcal{C}, \mathrm{id})"'] & \mathrm{GenCyc}(\mathcal{E}) \arrow[d] \\ \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \arrow[r, "\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}"] & \mathrm{GenPgc}(\mathcal{E}) \end{tikzcd}$$

Moreover, this cyclotomic refinement is such that $F(\mathcal{C}) \longrightarrow \mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}(\mathcal{C}, \mathrm{id})$ is a map of genuine cyclotomic objects where the source is given the trivial cyclotomic structure. In particular, we have a natural map

$$F(\mathcal{C}) \longrightarrow \mathrm{TC}(\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}(\mathcal{C}, \mathrm{id}))$$

in $\mathcal{E}$, which we call the *cyclotomic trace.*

Applied to K-theory, we therefore get a factorization of the trace map $\mathrm{K} \to \mathrm{THH}$ as

$$\mathrm{K}(\mathcal{C}) \longrightarrow \mathrm{TC}(\mathrm{THH}(\mathcal{C})) \longrightarrow \mathrm{THH}(\mathcal{C})$$

For bounded below cyclotomic spectrum, this recovers the cyclotomic trace of say [NS17], and as in the polygonic case, our factorization is generally finer. Piecing together the different maps we have built, we get a commutative diagram of spectra natural in $(\mathcal{C}, M) \in \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$:

$$\begin{tikzcd}
 & \mathrm{TR}(\mathcal{C}, M) \arrow[r] & \mathrm{THH}(\mathcal{C}, M) \\
\mathrm{K}(\mathrm{Lace}(\mathcal{C}, M)) \arrow[r] \arrow[d] \arrow[ur] & \mathrm{TC}(\mathrm{Lace}(\mathcal{C}, M)) \arrow[r] \arrow[d] \arrow[u] & \mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), \mathrm{id}) \arrow[d] \arrow[u] \\
\mathrm{K}(\mathcal{C}) \arrow[r] & \mathrm{TC}(\mathcal{C}) \arrow[r] & \mathrm{THH}(\mathcal{C}, \mathrm{id})
\end{tikzcd}$$

The goal of section §11 and the ultimate goal of this document is to explain why the bottom left square is cartesian. For this, it suffices to show that the induced map on vertical fibers, for which we have used the names $\mathrm{K}^{\mathrm{cyc}}$ and $\mathrm{TC}^{\mathrm{cyc}}$ respectively, is an equivalence. Note that this fiber is split and the map to $\mathrm{TR}(\mathcal{C}, M)$ factors through them since $\mathrm{TR}(\mathcal{C}, 0) \simeq 0$, so that we have a commutative triangle:

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, M) \arrow[r] \arrow[dr] & \mathrm{TC}^{\mathrm{cyc}}(\mathcal{C}, M) \arrow[d] \\
 & \mathrm{TR}(\mathcal{C}, M)
\end{tikzcd}$$

In the next sections, we will investigate the vertical maps and try to understand when they are equivalences: this is sufficient to try to prove that the square is indeed cartesian.

## 7.3 Higher derivatives and the polygonic structure

We have produced a lot of structure on the first derivative of a laced invariant. As foretold, this structure recovers the whole Taylor tower as we now explain. For this, we will need to introduce truncated versions of the genuine polygonic spectra; recall Definition 6.25 for the meaning of truncation sets.

**Proposition 7.26** Let $\mathcal{E}$ be presentable stable. There is a contravariant functor $T \mapsto \mathrm{GenPgc}_T(\mathcal{E})$ sending a truncation set to the category of $T$-truncated polygonic spectra, where maps of truncations sets are given by inclusions.

For $T$ a truncation set, the presentable stable $\mathrm{GenPgc}_T(\mathcal{E})$ admits a colimit-preserving functor $\mathrm{triv}_T : \mathcal{E} \to \mathrm{GenPgc}_T(\mathcal{E})$ compatible with the functoriality, as well as a conservative functor

$$\mathrm{GenPgc}_T(\mathcal{E}) \longrightarrow \prod_{t \in T} \mathcal{E}^{BC_t}$$

We write $\mathrm{TR}_T$ for the right adjoint to $\mathrm{triv}_T$.

**Proposition 7.27** Let $T \subset T'$ an inclusion of truncation sets such that the complement $T' - T = \{n\}$ is reduced to one element. For $X \in \mathrm{GenPgc}_{T'}(\mathcal{E})$, there is a natural exact square

$$\begin{tikzcd}
\mathrm{TR}_{T'}(X) \arrow[r] \arrow[d] & X_n^{\mathrm{hC}_n} \arrow[d, "\mathrm{can}"] \\
\mathrm{TR}_T(X) \arrow[r] & X_n^{\mathrm{tC}_n}
\end{tikzcd}$$

where we also denoted $X$ its image in $\mathrm{GenPgc}_T(\mathcal{E})$. Moreover, there is an equivalence

$$\mathrm{TR} \simeq \lim_{n \in (\mathbb{N}, |)^{\mathrm{op}}} \mathrm{TR}_{[n]}$$

where $[n]$ is the truncation set $\{0, 1, ..., n\}$ and we omitted to write the restriction functors.

Comparing the above square with the usual Kuhn-McCarthy square of Goodwillie calculus (or the version adapted to the fiberwise-exact situation such as in section 6), we get the following improvement of the main result of [LM12]:

**Theorem 7.28** Let $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ be a finitary Verdier-localizing invariant. Then, there is a canonical equivalence

$$\mathrm{P}_n^{\mathrm{fbw}} F^{\mathrm{lace}} \xrightarrow{\simeq} \mathrm{TR}_{[n]}(\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})$$

where $\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}}$ is endowed with the genuine polygonic structure of Theorem 7.18. In particular, at the limit, we get that the map

$$F^{\mathrm{lace}} \longrightarrow \mathrm{TR}(\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}})$$

identifies its target with the limit of the Taylor tower of $F^{\mathrm{lace}}$.

**Corollary 7.29** Let $\eta : F \to G$ be a natural transformation of Verdier-localizing invariants. Suppose that $\eta$ induces an equivalence

$$\mathrm{P}_1^{\mathrm{fbw}} F^{\mathrm{lace}} \xrightarrow{\simeq} \mathrm{P}_1^{\mathrm{fbw}} G^{lace}$$

Then $\eta$ identifies the Taylor tower of $F^{\mathrm{lace}}$ and $G^{lace}$.

**Remark 7.30** Let us make a historical remark: the Dundas-Goodwillie-McCarthy theorem, the main result of [DGM13], predates the computation of the Taylor tower of K-theory by Lindenstrauss-McCarthy [LM12] (even though the dates of publications seem to tell a different story). The reason this is possible is precisely the phenomenon of Corollary 7.29 which was identified by Goodwillie, but under assumptions of analycity — i.e. convergence of the Taylor tower. In spirit, this is also how Raskin describes the proof in [Ras18].

Our proof is quite different, and thus more than just a simple upgrade of the usual proof to a more modern context. In the above section, it was shown that the computation of the Taylor tower is independent of any convergence phenomenon. Since they are understood prior to convergence, we will further insist on the higher derivatives when studying analycity so that our arguments can be stated more generally than if we tried to bootstrap the phenomenon of Corollary 7.29.

# 8 Lacing topological Hochschild homology, topological cyclic homology

The previous sections have amounted to two things: we have a wonderful gadget, which takes a Verdier localizing functor $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$ and produces a functor $\mathrm{P}_1^{\mathrm{fbw}} F : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{GenPgc}(\mathcal{E})$ valued in genuine polygonic objects, whose restriction to $\mathbf{Cat}^{\mathrm{Ex}}$ is a functor

$$\mathrm{P}_1^{\mathrm{fbw}} F : \mathbf{Cat}^{\mathrm{Ex}} \longrightarrow \mathrm{GenCyc}(\mathcal{E})$$

valued in genuine cyclotomic objects. We have also ran this machine on a concrete example, namely the K-theory functor $\mathrm{K} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ and got as a result the genuine cyclotomic structure on $\mathrm{THH} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$, which in particular produces a Verdier-localizing functor $\mathrm{TC} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$.

There is no point in developing this much theory to only apply it once, so the goal of this section is to play again, with the new functors we produced. As tempting as it is to work with TC, the approach we choose is to actually feed THH with its extra structure to the theory.

## 8.1 The derivatives of laced THH

Thanks to Theorem 7.25, we have built a Verdier-localizing (in fact Karoubi-localizing) $\mathrm{THH} : \mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{GenCyc}(\mathrm{Sp})$. In particular, we can consider the following functor on the tangent, obtained by lacing it:

$$\begin{aligned} \mathrm{THH}^{\mathrm{lace}} : \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}} &\longrightarrow \qquad GenCyc(\mathrm{Sp}) \\ (\mathcal{C}, M) &\longmapsto \mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), \mathrm{id}) \end{aligned}$$

As we did in K-theory, we begin by a computation of the first derivative. There is a natural map

$$\mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), \mathrm{id}) \longrightarrow \mathrm{THH}(\mathcal{C}, M)$$

induced by the counit of the $L := (-, \mathrm{id}) \dashv \mathrm{Lace}$ adjunction. It induces a $S^1$-equivariant natural map map

$$\alpha(\mathcal{C}, M) : \mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), \mathrm{id}) \longrightarrow \mathrm{coind}_*^{S^1} \mathrm{THH}(\mathcal{C}, M)$$

where $\mathrm{coind}_*^{S^1} : \mathrm{Sp} \to \mathrm{Sp}^{BS^1}$ is the right adjoint to the forgetful functor. In particular, it is also the right Kan extension along the inclusion $* \to BS^1$ and given by the formula $\mathrm{Map}(S^1, -)$ with $S^1$-action given by precomposition. In consequence, we can also write an equalizer formula

$$\mathrm{coind}_*^{S^1} \mathrm{THH}(\mathcal{C}, M) \simeq \mathrm{Eq}\left( \mathrm{THH}(\mathcal{C}, M) \rightrightarrows \mathrm{THH}(\mathcal{C}, M) \right)$$

because $S^1$ is the suspension of $S^0$, i.e. the coequalizer of two nullhomotopic maps from $S^0$ to itself.

In fact, we can promote this natural transformation to a map of (genuine) cyclotomic spectra. Abstractly, this is clear because the forgetful functor $\mathrm{GenCyc}(\mathrm{Sp}) \to \mathrm{Sp}$ factors through the forgetful functor $\mathrm{Sp}^{BS^1} \to \mathrm{Sp}$ and every functor in sight has a right adjoint, which therefore must compose well.

We still write $\mathrm{coind}_*^{S^1}$ for the right adjoint to $\mathrm{GenCyc}(\mathrm{Sp})$ and we claim that the underlying spectrum of this right adjoint coincides given by the above equalizer; in fact, $\mathrm{coind}_*^{S^1} X$ is actually a genuine cyclotomic spectrum with Frobenius lifts and the maps to homotopy fixed points are induced by the fact that the $S^1$-action is trivial.

**Proposition 8.1** Let $(\mathcal{C}, M)$ be a laced category. Then the natural transformation $\alpha$ induces an equivalence of first derivatives. In particular,

$$(\mathrm{P}_1^{\mathrm{fbw}} \mathrm{THH}^{\mathrm{lace}})(\mathcal{C}, M) \xrightarrow{\simeq} \mathrm{coind}_*^{S^1} \mathrm{THH}(\mathcal{C}, M)$$

is an equivalence of genuine cyclotomic spectra where the right hand side has the aforementioned structure.

**Proof.** It suffices to check that the map of spectra is an equivalence. Write $p : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}$ and $R : \mathrm{Ind}\,\mathcal{C} \to \mathrm{Ind}\,\mathrm{Lace}(\mathcal{C}, M)$ for the Ind-right adjoint of $p$. We have

$$\mathrm{map}_{\mathrm{Lace}(\mathcal{C},M)} := \mathrm{Eq}\left( (p^{\mathrm{op}} \times p)^* \mathrm{map}_{\mathcal{C}} \rightrightarrows (p^{\mathrm{op}} \times p)^* M \right)$$

in the category of $\mathrm{Lace}(\mathcal{C}, M)$-bimodules, Since THH is exact in the $M$-variable, it commutes with the above equalizer, so that the following formula:

$$\mathrm{Eq}\left( \mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), (p^{\mathrm{op}} \times p)^* \mathrm{map}_{\mathcal{C}}) \rightrightarrows \mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), (p^{\mathrm{op}} \times p)^* M) \right)$$

computes $\mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), \mathrm{map}_{\mathrm{Lace}(\mathcal{C},M)})$. Now, using the functors to the Ind perspective on bimodules, we can rewrite:

$$\mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), (p^{\mathrm{op}} \times p)^* M) = \mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), R \circ M \circ p)$$

so that it becomes clear that the cyclic-invariance of THH implies the following:

$$\mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M), \mathrm{map}_{\mathrm{Lace}(\mathcal{C},M)}) := \mathrm{Eq}\left( \mathrm{THH}(\mathcal{C}, p \circ R) \rightrightarrows \mathrm{THH}(\mathcal{C}, p \circ R \circ M) \right)$$

Since $\mathrm{THH}(\mathcal{C}, -)$ is actually colimit-preserving, it suffices to compute the derivative in $M$ of the functors $M \mapsto p \circ R$ and $M \mapsto p \circ R \circ M$ (note that $p \circ R$ does depend on $M$ even though the notation does not show it).

We have already done this computation during the proof of Theorem 6.18, and we remark that the reduced part of $p \circ R$ was the constant functor with value the identity $\mathrm{id}_{\mathrm{Ind}\,\mathcal{C}}$ and the first derivative exactly $M \mapsto M$. In particular, we get

$$\mathrm{P}_1^{\mathrm{fbw}} \mathrm{THH}^{\mathrm{lace}}(\mathcal{C}, M) := \mathrm{Eq}\left( \mathrm{THH}(\mathcal{C}, M) \rightrightarrows \mathrm{THH}(\mathcal{C}, M) \right)$$

which is exactly what we wanted. ☐

**Remark 8.2** Note that the above proof works more generally for all cyclic-invariant which are colimit-preserving in the bimodule. However, they are all of the form $X \otimes \mathrm{THH}$ by a theorem of Ramzi [Ram24b] so the above proof is already almost at its maximal degree of generality.

We now turn to the higher derivatives. Using Theorem 7.18, we are reduced to understand the polygonic structure on $\mathrm{coind}_*^{S^1} \mathrm{THH}(\mathcal{C}, M)$. Our first remark, which is dual to [LT23, Lemma 4.26], is the following:

**Lemma 8.3** Let $X$ be a $\mathrm{C}_n$-spectrum. Then, $Y := \mathrm{coind}_*^{S^1}(X) \simeq \mathrm{Map}(S^1, X)$ acquires the structure of an $(S^1 \times \mathrm{C}_n)$-spectrum where $\mathrm{C}_n$ acts by conjugating the actions on $S^1$ by rotation and on $X$.

It holds that $Y$ is in the smallest stable subcategory of $(S^1 \times \mathrm{C}_n)$-spectra generated by free $\mathrm{C}_n$-spectrum i.e. whose $\mathrm{C}_n$-action is induced from $*$. Hence, $Y^{\mathrm{tC}_n} \simeq Y^{\tau \mathrm{C}_n} \simeq 0$ and consequently, there is a $S^1$-equivariant equivalence:

$$\left(\mathrm{coind}_*^{S^1} X\right)_{\mathrm{hC}_n} \xrightarrow{\simeq} \mathrm{coind}_{\mathrm{C}_n}^{S^1} X$$

where $\mu_n$ denotes the canonical copy of $\mathrm{C}_n$ in $S^1$.

**Proof.** Since $(S^1)_{\mathrm{hC}_n} \simeq S^1/\mathrm{C}_n$, we have:

$$\left(\mathrm{coind}_*^{S^1} X\right)^{\mathrm{hC}_n} \simeq \mathrm{Map}(S^1, X)^{\mathrm{hC}_n} \simeq \mathrm{Map}(S^1/\mathrm{C}_n, X) \simeq \mathrm{coind}_{\mu_n}^{S^1} X$$

Thus the last claim reduces to the first. Moreover, we can realize $S^1$ as the following coequalizer of spaces:

$$S^1 \simeq \mathrm{CoEq}\left( \{1, ..., n\} \overset{\mathrm{id}}{\underset{+1 \bmod n}{\rightrightarrows}} \{1, ..., n\} \right)$$

where $\{1, ..., n\}$ is the discrete space with $n$ points. Passing to $\mathrm{Map}(-, X)$ we get the following equalizer in Sp:

$$\mathrm{coind}_*^{S^1} X = \mathrm{Map}(S^1, X) \simeq \mathrm{Eq}\left( X^{\oplus n} \overset{\mathrm{id}}{\underset{\omega_n}{\rightrightarrows}} X^{\oplus n} \right)$$

where $\omega_n$ shifts cyclicly each copy of $X$. Note that with respect to the conjugation action, both maps are $\mathrm{C}_n$-equivariant so this is a $\mathrm{C}_n$-equivariant equalizer. It follows that $\mathrm{coind}_*^{S^1} X$ is in the smallest stable subcategory generated by $X^{\oplus n} \simeq \mathrm{ind}_*^{\mathrm{C}_n} X$. ☐

In particular, $\mathrm{P}_1^{\mathrm{fbw}} \mathrm{THH}^{\mathrm{lace}}$ is of the above form. This means that there will not be any non-trivial Tate constructions and thus any non-trivial maps to supply in the polygonic part of the structure; to be thorough, we would need to justify that the cyclic action of Theorem 7.18 is indeed of the above form. We will not show this as it would imply figuring out how the proof of the cited Theorem works; instead, we cite [HNS25] once again (and for the last time).

**Theorem 8.4** The functor $\mathrm{P}_1^{\mathrm{fbw}} \mathrm{THH}^{\mathrm{lace}} : \mathbf{TCat}^{\mathrm{Ex}} \to \mathrm{GenPgc}(\mathrm{GenCyc}(\mathrm{Sp}))$ lands in the subcategory spanned by $X_n$ such that $X_{nk}^{\tau \mathrm{C}_k} \simeq 0$. In particular, the $n^{th}$-derivative of $\mathrm{THH}^{\mathrm{lace}}$ has underlying spectrum given by:

$$(\mathrm{P}_n^{\mathrm{fbw}} \mathrm{THH}^{\mathrm{lace}})(\mathcal{C}, M) \simeq \bigoplus_{1 \leq k \leq n} \mathrm{coind}_{\mu_k}^{S^1} \mathrm{THH}(\mathcal{C}, M^{\otimes k})$$

Moreover, as a genuine cyclotomic spectrum, this expression coincides with $R_{[n]}(\mathrm{THH}(\mathcal{C}, M))$ where THH is given its truncated genuine polygonic structure and $R_{[n]}$ is the right adjoint of the composite forgetful functor $\mathrm{GenCyc}(\mathrm{Sp}) \to \mathrm{GenPgc}(\mathrm{Sp}) \to \mathrm{GenPgc}_{[n]}(\mathrm{Sp})$, built in Proposition 7.22 and Proposition 7.26.

In consequence, the limit of the Taylor tower of $\mathrm{THH}^{\mathrm{lace}}$ is given by

$$\mathrm{THH}^{\mathrm{lace}}(\mathcal{C}, M) \longrightarrow R\,\mathrm{THH}(\mathcal{C}, M)$$

where $R$ is the right adjoint to $\mathrm{GenCyc}(\mathrm{Sp}) \to \mathrm{GenPgc}(\mathrm{Sp})$.

**Remark 8.5** In general, it need not be that $R$ is given by an infinite product similar to the previous formula, because non-finite limits are not computed underlying in either GenCyc(Sp) or GenPgc(Sp) (more precisely, it is an infinite product but only in the category GenCyc(Sp) and thus not underlying).

However, this holds when all the underlying spectra are uniformly bounded-below: in fact in this case, since the difference between and non-genuine disappears, we can directly cite [KMN23, Corollary 2.27].

## 8.2 Laced topological cyclic homology

The astute reader will certainly expect at this point that to understand the following functor

$$\mathrm{TC}^{\mathrm{lace}}(\mathcal{C}, M) := \mathrm{TC}(\mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M)))$$

where THH is given its genuine cyclotomic structure, we would follow the same route and compute derivatives. But actually, the previous section is already enough, astuteness be damned! In fact, computing derivatives of TC would be annoying because $\mathrm{TC} : \mathrm{GenCyc}(\mathrm{Sp}) \to \mathrm{Sp}$ is not filtered-colimit preserving and we do not know to get them in full generality, so we are happy to have saved ourselves the trouble. Recall the following observation:

**Lemma 8.6** Let $X$ be a genuine polygonic spectrum. There is an equivalence

$$\mathrm{TC}(R(X)) \simeq \mathrm{TR}(X)$$

where $R : \mathrm{GenPgc}(\mathrm{Sp}) \to \mathrm{GenCyc}(\mathrm{Sp})$ is the right adjoint of the canonical functor.

In particular, in light of Theorem 8.4, we see that:

**Corollary 8.7** Let $(\mathcal{C}, M)$ be a laced-category. There is a natural map

$$\mathrm{TC}^{\mathrm{cyc}}(\mathcal{C}, M) := \mathrm{fib}(\mathrm{TC}^{\mathrm{lace}}(\mathcal{C}, M) \to \mathrm{TC}(\mathcal{C})) \longrightarrow \mathrm{TR}(\mathcal{C}, M)$$

Moreover, this map is an equivalence as soon $\mathrm{fib}(\mathrm{THH}^{\mathrm{lace}}(\mathcal{C}, M) \to \mathrm{THH}(\mathcal{C}))$ coincides with the limit of the Taylor tower.

In particular, we have a commutative diagram

$$\begin{tikzcd} \mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, M) \arrow[r] \arrow[dr] & \mathrm{TC}^{\mathrm{cyc}}(\mathcal{C}, M) \arrow[d] \\ & \mathrm{TR}(\mathcal{C}, M) \end{tikzcd}$$

and we have reduced the study of when the horizontal map is an equivalence to understanding when two functors coincide with their Taylor towers, namely $\mathrm{K}^{\mathrm{lace}}$ and $\mathrm{THH}^{\mathrm{lace}}$.

# 9 A general criterion for analytic functors

This section is an interlude: nothing here specifically talks about K-theory or THH, although those ideas have been introduced due to trace methods considerations by Goodwillie in the series of articles [Goo90, Goo91, Goo03]. Here, we are particularly interested in the second article and notably, in the concept of analytic functors, which we revisit here. Some of the results are standard folklore but we do not of a reference, or even whether Theorem 9.19 is folklore.

## 9.1 Total fiber of cubes

We begin by a discussion about cubes and their total fibers, following ideas of Section 1 of [Goo91]. After writing this section, the author was made aware of [ACB22] whose section 2 also proves many of the results which follow. First, let us work through how it works for 2-cubes. We fix a stable category $\mathcal{C}$ and the following square therein:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \tag{$\square$}$$

**Lemma 9.1** There is a canonical equivalence

$$\mathrm{fib}(\mathrm{fib}(A \to C) \longrightarrow \mathrm{fib}(B \to D)) \simeq \mathrm{fib}(\mathrm{fib}(A \to B) \longrightarrow \mathrm{fib}(C \to D))$$

identifying the fiber of the map between vertical fibers and the fiber of the map between horizontal fibers.

**Proof.** The following is a diagram of exact rows and columns — except one a priori, but since $\mathcal{C}$ is stable, it satisfies the $3 \times 3$ lemma hence every horizontal and vertical sequence below is exact:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{fib}(A \to C) \arrow[r] \arrow[d] & \mathrm{fib}(B \to D) \arrow[d] \\ \mathrm{fib}(A \to B) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[d] \\ \mathrm{fib}(C \to D) \arrow[r] & C \arrow[r] & D \end{tikzcd}$$

In particular, the top left vertex is the common fiber as wanted. $\square$

**Lemma 9.2** Denote $P := C \times_D B$ the pullback of $\square$. Then, there there is a canonical fiber sequence

$$\mathrm{fib}(A \to P) \longrightarrow \mathrm{fib}(A \to C) \longrightarrow \mathrm{fib}(B \to D)$$

Differently stated, the common fiber of Lemma 9.1 is also the fiber of $A \to P$.

**Proof.** Let $X \in \mathcal{C}$, then a map $X \to \mathrm{fib}(A \to C) \longrightarrow \mathrm{fib}(B \to D)$ is equivalently given by a diagram of the following shape:

$$\begin{tikzcd} X \arrow[dr] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "0"'] & & \\ & A \arrow[r] \arrow[d] & B \arrow[d] \\ & C \arrow[r] & D \end{tikzcd}$$

where the 2-simplex making the outer square commute is also zero. This is equivalent to the datum of diagrams $X \to A \to P$ with a nullhomotopy of the composite. This concludes. $\square$

The previous phenomenon is not limited to squares: it also holds for higher cubes. Recall that a $n$-cube is a functor $\mathcal{P}([n]) \to \mathcal{C}$ where $\mathcal{P}([n])$ denotes the poset of subsets of $[n]$.

**Definition 9.3** Let $X$ be a cube. The total fiber of $X$ is the fiber of the canonical map:

$$X(\emptyset) \longrightarrow \lim_{S \neq \emptyset} X_S$$

■ **Example 9.4** A 1-cube is a map $A \to B$ and its total fiber is simply the fiber of the map. ■

There are exactly $n+1$ functors $\mathcal{P}([n]) \to \mathcal{P}([n+1])$ corresponding to the poset inclusions $[n] \to [n+1]$, which induce $n+1$ equivalences $\mathcal{P}([n]) \times \mathcal{P}([1]) \simeq \mathcal{P}([n+1])$, corresponding to the choice of a direction, showing that a $(n+1)$-cube is nothing else than a map of $n$-cube.

In particular, we have shown previously that, independently of the direction, the total fiber of a 2-cube (more commonly known as a *square*) can also be computed by taking its honest fiber as a map of 1-cubes and then the total fiber of the resulting 1-cube.

**Theorem 9.5 — Goodwillie.** Let $X$ be a $(n+1)$-cube and consider $X_s \to X_t$ any of the associated map of $n$-cubes. Then, we have a canonical fiber sequence

$$\mathrm{totfib}(X) \longrightarrow \mathrm{totfib}(X_s) \longrightarrow \mathrm{totfib}(X_t)$$

In particular, the fiber of the right hand map does not depend of a choice of left and right.

**Proof.** The result for spaces appears as Definition 1.1 in[Goo91]. More generally, we can simply adapt the argument of Lemma 9.2 to $(n+1)$-cubes. Without loss of generality, we pick the map $[n] \to [n+1]$ which misses $(n+1)$; we have a commutative square:

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & \lim\limits_{\substack{S\subset[n] \\ S\neq\emptyset}} X_S \arrow[d] \\ X_{\{n+1\}} \arrow[r] & \lim\limits_{\substack{S\subset[n] \\ S\neq\emptyset}} X_{S\cup\{n+1\}} \end{tikzcd}$$

Remark that $\mathrm{totfib}(X_{left})$ is the fiber of the horizontal top map where as $\mathrm{totfib}(X_{right})$ is the fiber of the horizontal bottom map. Hence Lemma 9.2 concludes if we can show that the total fiber of the square is also $\mathrm{totfib}(X)$; this clearly reduces to showing that

$$\lim_{\substack{S\subset[n+1] \\ S\neq\emptyset}} X_S \longrightarrow X_{\{n+1\}} \underset{\left(\lim_{\substack{S\subset[n] \\ S\neq\emptyset}} X_{S\cup\{n+1\}}\right)}{\times} \lim_{S\subset[n]S\neq\emptyset} X_S$$

is an equivalence, which follows from usual decomposition rules on limits (see [Lur08, Proposition 4.4.2.2]). ☐

## 9.2 Analytic functors

We now record what we will mean by *analytic functors*. The notion was also introduced by Goodwillie in [Goo91], although in slightly different words. We use the notations introduced in §6.1, where we introduced (higher) Goodwillie calculus.

**Definition 9.6** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable stable categories. Denote $\mathrm{P}_\infty F$ the limit of the tower of $\mathrm{P}_n F$, there is a natural transformation $F \to \mathrm{P}_\infty F$.

We say that $F$ is *analytic* at $X \in \mathcal{C}$ if the map

$$F(X) \xrightarrow{\simeq} \mathrm{P}_\infty F(X)$$

is an equivalence. When $F$ is analytic for every $X$, then we will say that $F$ is analytic on $\mathcal{C}$, and if $F$ is only analytic for some subcategory $\mathcal{C}_0$, we will say that $F$ is $\mathcal{C}_0$-analytic.

We adopt similar, fiberwise definitions for functors defined on $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, replacing $\mathrm{P}_\infty$ by $\mathrm{P}_\infty^{\mathrm{fbw}}$.

**Remark 9.7 — Mind the gap.** In [Goo91], a functor $F$ is $\rho$-analytic ($\rho \in \mathbb{N}$) if it is equivalent to the limit of its Taylor tower on $(\rho+1)$-connected spaces.

The goal of this subsection is to give a pratical criterion for a functor to be analytic. We begin by remarking that if $\mathcal{C}$ is presentably stable symmetric monoidal and $A$ is a $\Sigma_n$-object in $\mathcal{C}$, $X \mapsto (A \otimes X^{\otimes n})_{\mathrm{h}\Sigma_n}$ is reduced, $n$-homogeneous and preserves sifted colimits (recall that $I$ is *sifted* if $I \to I^{\times k}$ is cofinal for $k \geq 1$). In particular, for any choice of $A_k$ with $\Sigma_k$-actions, the functor

$$X \longmapsto \coprod_{n\geq 1} (A_n \otimes X^{\otimes n})_{\mathrm{h}\Sigma_n}$$

preserves sifted colimits. In fact, in some favorable cases such as when $\mathcal{C}$ is the category of $T(n)$-local spectra, every sifted-colimit preserving functor is of this form (see [Lur17b, Lectures 8 and 9] where such functors are called *coanalytic*).

Remark that the above expression is a functor whose $n$-homogeneous part is given by $(A_n \otimes X^{\otimes n})_{\mathrm{h}\Sigma_n}$ and whose Goodwillie-Taylor tower is split with limit is given by the same expression but with a product. In good circumstances, for instance if $\mathcal{C}$ has a right-complete t-structure compatible with filtered colimits, the $A_n$ are connective and $X$ is taken to be 1-connective, then the product and the coproduct coincide so that this functor is actually analytic.

In fact, our criterion for analyticity will not be too different, but it will need to take into account a more general situation in two different ways. First, neither $\mathrm{K}^{\mathrm{lace}}$ nor $\mathrm{THH}^{\mathrm{lace}}$ preserve fiberwise sifted colimits globally, but only for sufficiently nice ("connective") objects. Secondly, the Taylor towers of these two functors are not split.

For the rest of this section, let $\mathcal{C}, \mathcal{D}$ be stable categories and suppose $\mathcal{D}$ has a right-complete t-structure compatible with filtered colimits. Fix also $\mathcal{C}_{\geq 0}$ a full subcategory closed under colimits of $\mathcal{C}$ whose objects we refer to as *connective*.

**Definition 9.8** A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *rigid* if it is reduced, sends connective objects to connective objects and preserves sifted colimits of connective objects.

**Warning 9.9** Even if we write $\mathcal{C}_{\geq 0}$ and call its objects connective, it need not be that this category is the connective part of a t-structure on $\mathcal{C}$. To further confuse the reader, we will write $\mathcal{C}_{\geq n} := \Sigma^n \mathcal{C}_{\geq 0}$ and call such objects $n$-connective.

However, we will need that $\mathcal{D}_{\geq 0}$ is the connective part of a t-structure on $\mathcal{D}$ which is right-complete, to be able to use the connectivity estimates of Dold-Kan as in [Lur17a, Proposition 1.2.4.5].

Our goal is to show that being rigid is a sufficient condition for a functor to be analytic on sufficiently connected objects. The following general lemma, stemming from Dold-Kan considerations, will be quite central for us.

**Lemma 9.10** Let $k \geq 1$ and $F : \mathcal{C} \to \mathcal{D}$ a reduced functor between stable categories. If $X \in \mathcal{C}$, the simplicial object $F(X^{\oplus \bullet})$ whose $n^{th}$-simplicies are given by $F(X \oplus ... \oplus X)$ with $n$ summands fits in an exact sequence:

$$|F(X^{\oplus \bullet})|_{\leq k-1} \longrightarrow |F(X^{\oplus \bullet})|_{\leq k} \longrightarrow \Sigma^k \mathrm{cr}_k F(X, ..., X)$$

where $\mathrm{cr}_k F$ is the $k^{th}$-cross-effect of $F$ and $|\cdot|_{\leq k}$ is the truncated geometric realization, i.e. the colimit of the simplicial object restricted to the full subcategory spanned by $[0], ..., [k]$.

**Proof.** Note that $X^{\oplus \bullet}$ is the simplicial object associated to the Čech nerve of the map $0 \to \Sigma X$, and $F(X^{\oplus \bullet})$ is obtained by post-composing this simplicial object by $F$. Moreover, since colimits are functorial, there is a canonical map

$$|F(X^{\oplus \bullet})|_{\leq k-1} \longrightarrow |F(X^{\oplus \bullet})|_{\leq k}$$

By [Lur17a, Remark 1.2.4.3], the wanted cofiber is precisely $\Sigma^k C_k$ where $C_k$ is the cofiber of

$$\operatorname*{colim}_{\substack{S \subset [k] \\ S \neq [k]}} F(X^{\oplus |S|}) \to F(X^{\oplus k})$$

This latter cofiber is $\mathrm{cr}_k F(X, ..., X)$ by [Lur17a, Proposition 6.3.3.13] (or by definition, depending on taste) which concludes. □

**Proposition 9.11** Let $F : \mathcal{C} \to \mathcal{D}$ be rigid and $X \in \mathcal{C}$ connective, then the map

$$f : F(X) \to \Omega F(\Sigma X)$$

is 1-connective (i.e. has a 1-connective cokernel). In particular, $F$ maps $n$-connective objects to $n$-connective objects for $n \geq 0$.

**Proof.** Let $X \in \mathcal{C}_{\geq 0}$. We show that $\Sigma F(X) \to F(\Sigma X)$ is 2-connected. The Čech nerve of the map $0 \to \Sigma X$ induces a map:

$$\left| \cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; X \oplus X \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; X \;\substack{\longrightarrow\\\longrightarrow}\; 0 \right| \xrightarrow{\simeq} \Sigma X$$

We note that this map is an equivalence: indeed, the truncated geometric realization $|X^{\oplus\bullet}|_{\leq 1}$ is already $\Sigma X$ by $k = 1$ in Lemma 9.10, and it also follows from this Lemma that the finite geometric realization coincides with $|X^{\oplus\bullet}|$ under the canonical map, since id is exact and thus all its $k$-cross-effect for $k \geq 2$ vanish (see Proposition 6.1.4.10 [Lur17a]).

Since $F$ preserves sifted colimits of connective objects and every $X^{\oplus n}$ is connective, this means $F(\Sigma X)$ can be computed as the geometric realization of $F(X^{\oplus\bullet})$. Remark that we still have

$$\Sigma F(X) \simeq |F(X^{\oplus\bullet})|_{\leq 1}$$

hence the map $\Sigma F(X) \to F(\Sigma X)$ is the canonical map

$$|F(X^{\oplus\bullet})|_{\leq 1} \to \operatorname*{colim}_n |F(X^{\oplus\bullet})|_{\leq n}$$

The wanted result now follows from either Lemma 9.10 by remarking that cross-effect are split-cofibers and thus $(\mathrm{cr}_k F)(X, ..., X)$ is still connective for $X$ connective or more directly [Lur17a, Proposition 1.2.4.5(4)], since $|F(X^{\oplus\bullet})|_{\leq 1} \to |F(X^{\oplus\bullet})|_{\leq n}$ is 2-connected for every $n$. □

For a rigid $F$ and a connective $X$, Lemma 9.10 shows that the biggest obstruction in connectivity to $F(X)$ being equivalent to $\mathrm{P}_1 F(X)$ is the second suspension of the second cross-effect $\mathrm{cr}_2 F(X, X)$. But $\mathrm{cr}_2 F$ is actually a functor in two variables, which is reduced and preserves sifted colimits in both. This allows for the following trick:

**Lemma 9.12** Suppose $F : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ is rigid separately in both variables and $n \geq 0$. Then, $F$ maps $n$-connective objects to $2n$-connective objects. More generally, if $F$ is rigid in $k$ variables, it maps $n$-connective objects to $kn$-connective objects.

**Proof.** Since $F$ is rigid, for any connective $X$, $F(X, -)$ sends $n$-connective objects to $n$-connective objects. But, then, for any $Y \in \mathcal{C}$ which is $n$-connective, we have a rigid functor which restricts to

$$F(-, Y) : \mathcal{C}_{\geq 0} \to \mathcal{D}_{\geq n}$$

Hence, this functor is again rigid when we consider $\mathcal{D}$ with the t-structure shifted by $n$ (where connective objects are $\mathcal{D}_{\geq n}$); indeed, connective objects in this new structure were connective before and preservation of connective sifted colimits is automatic. Then, applying Proposition 9.11 again concludes. □

In particular, thanks to Lemma 9.10 and Proposition 9.11, we have shown:

**Corollary 9.13** If $X$ is 1-connective and $F$ is rigid, then $F(X) \to \Omega F(\Sigma X)$ has a 3-connective cokernel, hence a 2-connective kernel.

Let us now attempt to climb up the Taylor tower. Let $X \in \mathcal{C}$, consider $S \mapsto C_S(X)$ the strongly cocartesian $n$-cube generated by the arrows $\{X \to 0\}$. As in [Lur17a, Construction 6.1.1.22], we write $T_n F(X)$ for the limit $\lim_{S \neq \emptyset} F(C_S(X))$. There is a map $f_n(X) : F(X) \to T_n F(X)$ and it is natural in $X$; in particular, $T_n F$ is a well-defined functor $\mathcal{C} \to \mathcal{D}$.

**Lemma 9.14** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $n \geq 2$. Then the total fiber $\mathcal{F}_n(X)$ of the $n$-cube $F(C_S(X))$, which is by definition the fiber of $F(X) \to T_n F(X)$, is given by:

$$\mathcal{F}_n(X) \simeq \mathrm{fib}\left(\mathcal{F}_{n-1}(X) \longrightarrow \Omega^{n-1}\mathrm{cr}_n F(\Sigma X, ..., \Sigma X)\right)$$

where $\mathrm{cr}_n F$ is the $n^{th}$-cross effect of $F$, i.e. the initial functor under $F(X_1 \oplus ... \oplus X_k)$ which is reduced in every of the $n$ variables.

**Proof.** Write $\mathcal{X}_n$ for the $n$-cube $F(C_S(X))$. Viewing $\mathcal{X}_n$ as a map of $(n-1)$-cubes in any way, its source is exactly $\mathcal{X}_{n-1}$, whose total fiber is $\mathcal{F}_{n-1}(X)$. Hence, by Theorem 9.5, it suffices to understand the target $(n-1)$-cube and compute its total fiber.

The target $(n-1)$-cube is a cube with 0 as its top vertex and $F(\Sigma X \oplus ... \oplus \Sigma X)$ with $k = |S|$ summands at the vertex $S \subset [n-1]$, and the maps are given by the canonical injections of the coproduct. It follows from [Lur17a, Proposition 6.3.3.13] that $\mathrm{cr}_n F(\Sigma X, ..., \Sigma X)$ identifies with the total fiber of the cube with value $F(\Sigma X \oplus ... \oplus \Sigma X)$ at the vertex $S \subset [n-1]$, where there are $(n-1) - |S|$ summands (in fact, for Goodwillie, this was a definition in [Goo03, Section 3]). Hence, the total fiber of the cube we want is exactly $\Omega^{n-1}\mathrm{cr}_n F(\Sigma X, ..., \Sigma X)$ which concludes. □

**Remark 9.15** Remark also that the following square has exact rows and columns:

$$\begin{tikzcd}
\Omega\widehat{D_n F}(X) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \widehat{D_n F}(X) \arrow[d] \\
\mathcal{F}_n(X) \arrow[r] \arrow[d] & F(X) \arrow[r] \arrow[d] & T_n F(X) \arrow[d] \\
\mathcal{F}_{n-1}(X) \arrow[r] & F(X) \arrow[r] & T_{n-1}F(X)
\end{tikzcd}$$

In particular, there is a fiber sequence

$$\mathcal{F}_n(X) \longrightarrow \mathcal{F}_{n-1}(X) \longrightarrow \widehat{D_n F}(X)$$

which, by the above proposition, identifies $\widehat{D_n F}(X)$ with $\Omega^{n-1}\mathrm{cr}_n F(\Sigma X, ..., \Sigma X)$.

In particular, the following holds:

**Lemma 9.16** Let $F : \mathcal{C} \to \mathcal{D}$ be rigid. Then $T_n F : \mathcal{C} \to \mathcal{D}$ is rigid.

**Proof.** Given the formula, it is clear that $T_n F$ is reduced. Since both $\mathcal{C}$ and $\mathcal{D}$ are stable, sifted colimits commute with finite limits so that the formula also shows that $T_n F$ preserves sifted colimits of diagrams valued in $\mathcal{C}_{\geq 0}$.

Finally, to show that $T_n F$ preserves connective objects, Remark 9.15 provides an exact sequence

$$\Omega^{n-1}\mathrm{cr}_n F(\Sigma X, ..., \Sigma X) \longrightarrow T_n F(X) \longrightarrow T_{n-1}F(X)$$

By Lemma 9.12, the left hand side is connective and by induction so is the right hand side, the initial case $n = 1$ was dealt already Proposition 9.11. This concludes because connective objects are closed under extension. □

We are almost ready to prove the main result of this section; we recall before that a folklore result about cross-effect. Recall that $\mathrm{cr}_n F$ is the functor in $n$-variables under $F(X_1 \oplus ... \oplus X_n)$ which reduced in $X_1, ..., X_n$.

**Lemma 9.17** Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor, $n, k \geq 2$ and $X_1, ..., X_{n+k-1} \in \mathcal{C}$ then

$$\mathrm{cr}_k F(\mathrm{cr}_n F(X_1, ..., X_{n-1}, -))(X_n, ..., X_{n+k-1}) \simeq \mathrm{cr}_{n+k-1} F(X_1, ..., X_{n-1}, X_n, ..., X_{n+k-1})$$

functorially in the $X_i$.

**Proof.** By definition, $\mathrm{cr}_{n+k-1}F$ is the reducification in each variable of the functor $G_{n+k-1}[F]$ given by

$$G_{n+k-1}[F](X_1, ..., X_{n+k-1}) := F(X_1 \oplus ... \oplus X_{n+k-1})$$

In particular, the wanted identification follows from remarking that

$$\begin{aligned} G_{n+k-1}[F](X_1, ..., X_{n+k-1}) &\simeq G_n(X_1, ..., X_n \oplus ... \oplus X_{n+k-1}) \\ &\simeq G_k[G_n(X_1, ..., X_{n-1}, -)](X_n, ..., X_{n+k-1}) \end{aligned}$$

and this is of course functorial in the $X_i$. □

If $F$ is a functor in $n$ variables, and $\vec{m_i} := (m_1, ..., m_n)$ a tuple of $n$ integers, we write $\mathrm{P}_{\vec{m_i}} F$ the initial functor under $F$ which is $m_i$-excisive in the position $i$, which exists by [Lur17a, Proposition

6.1.3.6]. The proof of the previous lemma, combined with the fact that reducification does not change the value of derivatives (Remark 6.1.3.19 of [Lur17a]), immediately adapts to the following statement, which is probably less-known (so in consequence, more folklorish?):

**Corollary 9.18** Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor, $n, k \geq 2$ and $X_1, ..., X_{n-1} \in \mathcal{C}$ and $\vec{m_i} := (m_1, ..., m_{n-1})$, then

$$\mathrm{cr}_k F((\mathrm{P}_{\vec{m_i}}\mathrm{cr}_n F)(X_1, ..., X_{n-1}, -)) \simeq (\mathrm{P}_{\vec{m_i'}}\mathrm{cr}_{n+k-1} F)(X_1, ..., X_{n-1}, -, ..., -)$$

where, on the right hand side, we also denoted $\vec{m_i}$ the tuple $(m_1, ..., m_{n-1}, 0, ..., 0)$.

We have now enough to show the following, which will be our main criterion to show that functors are analytic on a suitable subcategory:

**Theorem 9.19** Let $F : \mathcal{C} \to \mathcal{D}$ be rigid and $X$ 1-connective, then the following map is an equivalence:

$$F(X) \xrightarrow{\simeq} \mathrm{P}_\infty F(X)$$

Differently stated, $F$ is analytic on 1-connective objects.

**Proof.** Let $X$ be 1-connective. Since the t-structure at the target is assumed right-complete, it suffices to show that $F(X) \to \mathrm{P}_n F(X)$ has connectivity going to $+\infty$ as $n \to +\infty$. By iterating Lemma 9.16, we see that $\mathrm{P}_n F$ is again rigid, so up to replacing $F$ by $\mathrm{fib}(F \to \mathrm{P}_n F)$, it suffices to show that if $F$ is such that $\mathrm{P}_k F \simeq 0$ for $1 \leq k \leq n$, then $F(X)$ is $(n+1)$-connective.

We note that this hypothesis implies that $\mathrm{P}_{(1,...,1)}\mathrm{cr}_k F(X_1, ..., X_k) \simeq 0$ for $1 \leq k \leq n$ and $(X_1, ..., X_k)$ any tuple of 1-connective objects. Consequently, leveraging Lemma 9.12 for $k \geq n+1$, we get that $\mathrm{P}_{(1,...,1)}\mathrm{cr}_k F(X_1, ..., X_k)$ is $(n+1)$-connective regardless of $k \geq 1$. We now prove by descending induction that

$$\mathrm{cr}_k F(X_1, ..., X_k) \longrightarrow \mathrm{P}_{(1,...,1)}\mathrm{cr}_k F(X_1, ..., X_k)$$

has $(n+1)$-connective fiber and consequently, $\mathrm{cr}_k F(X_1, ..., X_k)$ is also $(n+1)$-connective. This is clear for $k \geq n+1$. We factor this map through partial derivatives of the form $\mathrm{P}_{\vec{\delta_i}}$ where $\vec{\delta_i}$ is a tuple of zeroes up to $i$ and then ones. Note that the map

$$\mathrm{P}_{\vec{\delta_i}}\mathrm{cr}_k F(X_1, ..., X_k) \longrightarrow \mathrm{P}_{\vec{\delta_{i+1}}}\mathrm{cr}_k F(X_1, ..., X_k)$$

is as in Proposition 9.11, the map of a functor to its first derivative. Hence, similarly, Lemma 9.10 proves that its cofiber is an (infinite) extension of shifted cross-effects and by Corollary 9.18 these are cross-effects of order at least $(k+1)$. But now, by the induction hypotheses, all of them are at least $(n+1)$-connective. The claim thus follows.

But now, if all the $X_i$ are equal to the 1-connective $X$ we introduced previously, the resulting connectivity estimate on $\mathrm{cr}_k F(X, ..., X)$ implies a connectivity estimate on the cofiber of

$$\Sigma F(X) \longrightarrow F(\Sigma X)$$

by the same strategy as in Proposition 9.11, using the Dold-Kan estimates of Lemma 9.10. Namely, it follows that the map

$$F(X) \longrightarrow \Omega F(\Sigma X) \simeq T_1 F(X)$$

has $(n+1)$-connective fiber. Finally, remark that if $F$ is rigid such that $\mathrm{P}_k F \simeq 0$ for $1 \leq k \leq n$, then so is $T_1 F$ so that iteratively, we get that the map $F(X) \to P_1 F(X) \simeq 0$ has $(n+1)$-connective fiber. Consequently, $F(X)$ itself is $(n+1)$-connective which concludes. $\square$

# 10 Resolution theorems and rigidity

In the previous section, we have a criterion for analyticity which involves preservation of connectivity and of certain sifted colimits. The first part is rather straightforward for $\mathrm{K}^{\mathrm{lace}}(\mathcal{C}, -)$ since it always lands in connective spectra but the second requires more thought. The goal of this section is to introduce a framework, which was first developed in [Sau23b], which we reproduce in part, and which is completed by [SW25], which in particular allows to define a subcategory $\mathrm{Bimod}(\mathcal{C})_{\geq 0}$ closed under colimits in $\mathrm{Bimod}(\mathcal{C})$ and to show laced K-theory preserves sifted colimits when restricted to such "connective" bimodules.

## 10.1 Heart structures and resolution

Recall the following definition, due to Quillen in [Qui73] for 1-categories and Barwick in [Bar16] for higher categories.

**Definition 10.1 — Definition 3.1 of [Bar15].** Let $\mathcal{C}$ be an additive category. An *exact structure* on $\mathcal{C}$ is the datum of two subcategories $\mathcal{C}_{inj}$ and $\mathcal{C}_{proj}$ containing all equivalences and whose arrows are denoted respectively $\hookrightarrow$ and $\twoheadrightarrow$, which we choose to call respectively *exact inclusions* and *exact projections*, subject to the following conditions:

- For any $X \in \mathcal{C}$, $0 \hookrightarrow X$ is an exact inclusion and $X \twoheadrightarrow 0$ an exact projection.
- Exact inclusions are stable under pushout against any map and exact projections under pullback against any map[a].
- Any square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

  is a pullback of a span with a leg in $\mathcal{C}_{inj}$ and the other in $\mathcal{C}_{proj}$ if and only if it is a pushout of a cospan with the same condition.

If $\mathcal{C}$ is an exact category, we will call *exact sequences* those squares as in the third bullet point where $Z = 0$.

[a] In particular, such pullbacks and pushouts are required to exist.

In particular, every additive category can be promoted to an exact category by considering direct sums inclusions as exact and direct sums projections are exact.

On the other hand, considering stable categories as exact categories where every map is an exact inclusion and an exact projection provides a fully-faithful functor $\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Exact}_\infty$. This functor admits a left adjoint by work of Klemenc [Kle23], which we call the *stable envelope* functor and denote Stab. Klemenc proved the following:

**Proposition 10.2 — Theorem 1 of [Kle23].** Let $\mathcal{E}$ be an exact category, then the canonical functor $\mathcal{E} \to \mathrm{Stab}(\mathcal{E})$ is fully-faithful and detects exact sequences, so that $\mathcal{E}$ is in particular closed under extensions.

**Warning 10.3** We note that Stab is not in general the stabilization of a category with finite limits, which has been always denote $\mathrm{Sp}(-)$ in this text. In fact exact categories need not have all finite limits, and conversely categories with finite limits need not be additive hence cannot have exact structures.

We are particularly interested by the behavior of $\mathrm{K}(\mathcal{E}) \to \mathrm{K}(\mathrm{Stab}(\mathcal{E}))$. This is not new behavior: since its inception in Quillen's work, there has been this idea that K can be equivalently defined by using the additive category $\mathrm{Proj}(R)^{ft}$ of projective finite type $R$-modules or the stable category $\mathrm{Perf}(R)$ of compact $R$-modules[11].

Our goal in this section is to show that Quillen's original idea for tackling this question stands the test of time. We are led to introduce the following higher categorical analogue of the conditions of [Qui73, Theorem 3]

**Definition 10.4** Let $i : \mathcal{A} \to \mathcal{C}$ be a fully-faithful functor whose image is closed under extensions. Suppose $\mathcal{C}$ is exact and endow $\mathcal{A}$ with the inherited structure. We say that $i$ is *resolving* if it

[11] Or, if one sticks to Quillen's set-up, the abelian category given as the heart of the t-structure on $\mathrm{Perf}(R)$, the ordinary category of finitely presented $R$-modules, which by a famous theorem of Barwick has the same K-theory

satisfies the following two additional properties:

(i) For every exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$ with $Y \in \mathcal{A}$, we have $X \in \mathcal{A}$

(ii) For every $Z \in \mathcal{C}$, there is an exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$ with $Y \in \mathcal{A}$.

We say $i$ is op-resolving if $i^{\mathrm{op}}$ is resolving.

In subsequent sections, we will show that K-theory and THH, suitably defined for exact categories, does invert resolving inclusions. Thus, the first sanity check (which also happens to be an important part of the story) is the following:

**Lemma 10.5** Let $i : \mathcal{A} \to \mathcal{C}$ be a resolving functor. Then, it induces an equivalence

$$i^* : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{A}, \mathcal{D})$$

for every stable $\mathcal{D}$. In particular, $i$ induces an equivalence $\mathrm{Stab}(\mathcal{A}) \xrightarrow{\simeq} \mathrm{Stab}(\mathcal{C})$.

**Proof.** The crux of the proof lies in the fact that those conditions force the values on $\mathcal{C}$ from the values on $\mathcal{A}$ for functors which preserve exact sequences.

Let $X \in \mathcal{C}$, then we define $\mathrm{Ex}(X)$ to be the category of exact sequences

$$A \hookrightarrow B \twoheadrightarrow X$$

where $A, B \in \mathcal{A}$. Note that Ex is not *a priori* a functor in $X$. However, denote $\mathrm{Null}(X)$ the category of sequences $A \longrightarrow B \longrightarrow X$ with a given null homotopy of their composite, where we no longer require the maps to be exact inclusions or exact projections. Remark that $\mathrm{Null}(X)$ is clearly a functor in $X$ as it is the fiber of the map

$$\mathrm{Fun}(\{\bullet_2 \longleftarrow \bullet_0 \longrightarrow \bullet_1\}, \mathcal{C}_{/X}) \longrightarrow \mathrm{Fun}(\{\bullet_2\}, \mathcal{C}_{/X}) \simeq \mathcal{C}_{/X}$$

We claim the inclusion $\mathrm{Ex}(X) \to \mathrm{Null}(X)$ is cofinal. By [Lur08, Theorem 4.1.3.1], we are reduced to checking that for every $B_0 \to X$ with $B_0 \in \mathcal{A}$, the category of factorizations

$$B_0 \longrightarrow B \twoheadrightarrow X$$

is weakly contractible. This category is nonempty: indeed, by (ii) there is a map $B \twoheadrightarrow X$ with $B \in \mathcal{A}$ and the induced $B_0 \oplus B \twoheadrightarrow X$ is an exact projection by Lemma [Bar15, Lemma 4.7]. Moreover, this category also admits products, given by the pullback $B \times_X B' \twoheadrightarrow X$ equipped with its canonical map from $B_0$. One checks that $B \times_X B' \in \mathcal{A}$ by closure under extension of $\mathcal{A}$ in $\mathcal{C}$ and that the map is indeed a projections since they are stable under pullback and composition. Consequently, the comma category in question is indeed weakly contractible and the map cofinal.

In consequence, if $F : \mathcal{A} \to \mathcal{C}$ is any functor, the following formula defines a functor $R(F)$ with source $\mathcal{C}$ and target $\mathcal{D}$:

$$R(F)(X) := X \longmapsto \operatorname*{colim}_{\mathrm{Ex}(X)} (\mathrm{cofib}(F(A) \to F(B))) \simeq \operatorname*{colim}_{\mathrm{Null}(X)} (\mathrm{cofib}(F(A) \to F(B)))$$

Since colimits are functorial, $R$ upgrades to a functor $\mathrm{Fun}(\mathcal{A}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ taking $F$ to $R(F)$. The canonical $F(B) \to \mathrm{cofib}(F(A) \to F(B))$ induces a natural map in $X$ as follows:

$$\operatorname*{colim}_{\mathrm{Null}(X)} F(B) \longrightarrow \operatorname*{colim}_{\mathrm{Null}(X)} (\mathrm{cofib}(F(A) \to F(B)))$$

If $X \in \mathcal{A}$, then $\mathrm{Null}(X)$ has a terminal object given by the sequence $0 \hookrightarrow X \twoheadrightarrow X$ hence both colimits evaluate to $F(X)$, which provides a natural equivalence $F \to i^* R(F)$. This equivalence is again natural in $F$. Moreover, suppose $G : \mathcal{C} \to \mathcal{D}$ is a functor, there is a map $\mathrm{cofib}(G(A) \to G(B)) \to G(X)$ for every object of $\mathrm{Null}(X)$, hence we have a natural transformation $R(i^* G) \to G$, which is itself natural in $G$.

Suppose further that $F : \mathcal{A} \to \mathcal{D}$ is exact, then we claim that $R(F)$ is also exact. We first show that $F$ sends exact sequences of $\mathrm{Ex}(X)$ to exact sequences, and then we deal with the more general case. Let

$$A \hookrightarrow B \twoheadrightarrow X \qquad\qquad A' \hookrightarrow B' \twoheadrightarrow X$$

be exact sequences in $\mathrm{Ex}(X)$. We have a diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r, equal] \arrow[d, hook] & A \arrow[d, hook] \\
A' \arrow[r, hook] \arrow[d, equal] & B' \times_X B \arrow[r, two heads] \arrow[d, two heads] & B \arrow[d, two heads] \\
A' \arrow[r, hook] & B' \arrow[r, two heads] & X
\end{tikzcd}$$

By closure under extension of $\mathcal{A}$, we have that $B' \times_X B \in \mathcal{A}$. Since $F$ is exact in $\mathcal{A}$, it sends every sequence save for the bottom horizontal and the right vertical ones to exact sequences in $\mathcal{D}$. By pasting, taking the pushout $F(B') \coprod_{F(B' \times_X B)} F(B)$ completes the diagram where we applied to $F$ to have exact rows and columns. We deduce from this the following equivalences:

$$\mathrm{cofib}(F(A) \to F(B)) \simeq F(B') \coprod_{F(B' \times_X B)} F(B) \simeq \mathrm{cofib}(F(A') \to F(B'))$$

This shows that the functor $\mathrm{Ex}(X) \to \mathcal{C}$ sending $(A \hookrightarrow B \twoheadrightarrow X)$ to $\mathrm{cofib}(F(A) \to F(B))$ inverts every arrow in $\mathrm{Ex}(X)$.

Since $\mathrm{Ex}(X)$ is cofinal in $\mathrm{Null}(X)$ which has an initial object, $\mathrm{Ex}(X)$ is contractible; it follows that the colimit over $\mathrm{Ex}(X)$ of $\mathrm{cofib}(F(A) \to F(B))$ is constant. In particular, for every $(A \hookrightarrow B \twoheadrightarrow X)$, the canonical map is an equivalence

$$\mathrm{cofib}(F(A) \to F(B)) \longrightarrow \operatorname*{colim}_{\mathrm{Ex}(X)} (\mathrm{cofib}(F(A') \to F(B')))$$

Consequently, using the natural equivalence $F \to i^* R(F)$ we constructed, we see that $R(F)$ sends the objects of $\mathrm{Ex}(X)$ to exact sequences.

We now deal with the general case and show $R(F)$ sends all the exact sequences in $\mathcal{C}$ to exact sequences in $\mathcal{D}$. Let $X \hookrightarrow Y \twoheadrightarrow Z$ be an exact sequence of $\mathcal{C}$ and let $A \hookrightarrow B \twoheadrightarrow Y$ be an exact sequence with $A, B \in \mathcal{A}$ as provided by hypothesis (i) and (ii) of Definition 10.4. Then, we have a diagram with exact rows and columns:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d] & A \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\
X \times_Z B \arrow[r, hook] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & Z \arrow[d, equal] \\
X \arrow[r, hook] & Y \arrow[r, two heads] & Z
\end{tikzcd}$$

where $X \times_Z B \in \mathcal{A}$ by (i). Applying $R(F)$ to this diagram, every sequence save for the bottom horizontal one is sent to an exact one in $\mathcal{D}$, hence this is also the case for the bottom horizontal one, which shows the wanted statement. Consequently, $R(F)$ is exact as wanted.

We have a well-defined functor $R : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{A}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})$ coming with a natural equivalence $\mathrm{id} \to i^* \circ R$. Moreover, if $G : \mathcal{C} \to \mathcal{D}$ is exact then the natural transformation $R(i^* G) \to G$ we have constructed earlier is an equivalence; indeed, we have shown previously that this is the case on $\mathcal{A}$ and for every $X \in \mathcal{C}$, there is an exact sequence

$$A \hookrightarrow B \twoheadrightarrow X$$

such that $A, B \in \mathcal{A}$; the result now follows from the exactness of both $G$ and $R(i^* G)$. Hence $R$ is an equivalence with inverse $i^*$, which proves the wanted claim. $\square$

Of course, if $i : \mathcal{A} \to \mathcal{C}$ is op-resolving, applying the above to $i^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$ for a stable $\mathcal{D}$ immediately implies the dual version:

**Corollary 10.6** Let $i : \mathcal{A} \subset \mathcal{C}$ be op-resolving. Then, the following functor

$$i^* : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{A}, \mathcal{D})$$

is an equivalence for every stable $\mathcal{D}$. In particular, $i$ induces an equivalence $\mathrm{Stab}(\mathcal{A}) \simeq \mathrm{Stab}(\mathcal{C})$.

To leverage the idea of resolving functors against our problem, we will introduce an additional condition, which aims at answering the converse problem: given a stable category $\mathcal{C}$, how do we know if it is the stable envelope of some exact category $\mathcal{E}$ (other than itself).

**Definition 10.7** Let $\mathcal{C}$ be a stable category. A *heart structure* on $\mathcal{C}$ is the datum of a pair of full subcategories $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ subject to the following conditions:

- (i) $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ are closed under extensions, $\mathcal{C}_{\geq 0}$ under finite colimits and $\mathcal{C}_{\leq 0}$ under finite limits.
- (iii)[a] For any $C \in \mathcal{C}$, there is an exact sequence

  $$X \longrightarrow C \longrightarrow \Sigma Y$$

  with $X \in \mathcal{C}_{\leq 0}$ and $Y \in \mathcal{C}_{\geq 0}$.

A *heart-exact* functor between heart structures is an exact functor $f : \mathcal{C} \to \mathcal{D}$ on the underlying categories which preserves the two full subcategories of the structure. We denote $\mathbf{Cat}^{\mathrm{Ex}}_{\heartsuit}$ the category of heart structures and heart functors between them.

[a]The name of this condition is intentional, we simply consider condition (ii) empty.

Heart structures are a weakening of weight structures, introduced by Bondarko in [Bon10] for triangulated categories and refined to stable categories by Sosnilo in [Sos19]. Weight structures on a stable category satisfy the same axioms with the addition of a number (ii), which states for every $P \in \mathcal{C}_{\geq 0}$ and $N \in \mathcal{C}_{\leq 0}$, the mapping spectra $\mathrm{map}(N, P)$ must be connective.

In *loc. cit.*, Sosnilo showed that taking the heart — the intersection of non-positive and non-negative categories — is a fully-faithful functor with target additive categories, whose essential image is the subcategory of *weakly-idempotent complete* additive categories. Here, recall that an additive category is weakly-idempotent complete if every split-projection has a fiber.

**Definition 10.8** We write $\mathcal{C}_{[i;j]}$ for the intersection $\Sigma^j \mathcal{C}_{\leq 0} \cap \Sigma^i \mathcal{C}_{\geq 0}$. The *heart* of a heart structure is the category $\mathcal{C}_{[0;0]}$.

We will say that a heart structure on $\mathcal{C}$ is *bounded* if

$$\mathcal{C} \simeq \bigcup_{n \geq 0} \mathcal{C}_{[n;n]}$$

i.e. every object of $\mathcal{C}$ is both a shift of an object in $\mathcal{C}_{\leq 0}$ and an other shift of an object in $\mathcal{C}_{\geq 0}$.

Let us first record the following fact, which holds for weight structures by [HSH21, Lemma 3.1.5] and whose proof is exactly the same:

**Lemma 10.9** Let $a \leq c \leq b$ be three integers, $X \in \mathcal{C}_{[a,b]}$ and $Y \longrightarrow X \longrightarrow \Sigma Z$ an exact sequence with $Y \in \mathcal{C}_{]-\infty,c]}$ and $Z \in \mathcal{C}_{[c,+\infty[}$.

Then, we have $Y \in \mathcal{C}_{[a,c]}$ and $Z \in \mathcal{C}_{[c,b]}$.

**Proof.** This is the same proof as [HSH21, Lemma 3.1.5] which shows the corresponding fact for weight structures; consider the rotated exact sequences:

$$Z \longrightarrow Y \longrightarrow X \qquad\qquad X \longrightarrow \Sigma Z \longrightarrow \Sigma Y$$

The Lemma follows from closure under extensions of the relevant categories, which holds by hypothesis for us, applied to the above sequences. $\square$

**Theorem 10.10** Let $\mathcal{C}$ be a bounded heart category. Then, the inclusion $\mathcal{C}^\heartsuit \to \mathcal{C}$ factors as the (possibly infinite) composition of resolving or op-resolving functors. Consequently, we have an equivalence

$$\mathrm{Stab}(\mathcal{C}^\heartsuit) \xrightarrow{\simeq} \mathcal{C}$$

**Proof.** The result follows from the following two facts:

(a) Let $i \leq j$, then $\mathcal{C}_{[i,j]} \to \mathcal{C}_{[i,j+1]}$ is resolving

(b) Let $i \leq j$, then $\mathcal{C}_{[i,j]} \to \mathcal{C}_{[i-1,j]}$ is op-resolving

Since the heart structure on $\mathcal{C}$ is bounded, this implies that the inclusion $\mathcal{C}^\heartsuit \to \mathcal{C}$ is a composition of resolving and op-resolving functors as wanted. The proof of (b) will be dual to the proof of (a), hence let us only do the latter. We remark already that $\mathcal{C}_{[i,j]}$ is closed under extensions in $\mathcal{C}$ hence in all of its exact subcategories.

Let $X \hookrightarrow Y \twoheadrightarrow Z$ be an exact sequence such that $X, Z \in \mathcal{C}_{[i,j+1]}$ and $Y \in \mathcal{C}_{[i,j]}$. Then, Lemma 10.9 ensures that $X \in \mathcal{C}_{[i,j]}$ which gives (i).

If $Z \in \mathcal{C}_{[i,j+1]}$, then applying condition (iii) of Definition 10.7 to $\Omega^j Z$ implies there is an exact sequence

$$X \hookrightarrow Y \twoheadrightarrow Z$$

such that $X \in \mathcal{C}_{\geq j}$ and $Y \in \mathcal{C}_{\leq j}$. It follows from Lemma 10.9 that $Y \in \mathcal{C}_{[i,j]}$, which gives (ii). $\square$

In section §11.3, we will provide examples of interesting heart structures appearing "in the wild". We delay the concrete application of the above statement until then, and instead, deduce the abstract consequence in what follows.

**Corollary 10.11** The functor $(-)^\heartsuit : \mathbf{Cat}^{\mathrm{Ex}}_{\heartsuit,b} \to \mathbf{Exact}_\infty$ taking a bounded heart category to its heart is fully-faithful.

**Proof.** Let $\mathcal{C}, \mathcal{D}$ be heart categories, we want to prove that the functor

$$\Phi : \mathrm{Fun}^{\heartsuit-\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^\heartsuit, \mathcal{D}^\heartsuit)$$

is an equivalence, the left hand side denoting the full subcategory of $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D})$ spanned by heart-exact functors. By Theorem 10.10, the inclusion $i : \mathcal{C}^\heartsuit \to \mathcal{C}$ induces an equivalence:

$$i^* : \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^\heartsuit, \mathcal{D})$$

Under this equivalence, the full subcategory of heart-exact functors is mapped to a full subcategory of $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}^\heartsuit, \mathcal{D}^\heartsuit)$. One checks that by the explicit formula for the inverse provided in Lemma 10.5 that any functor $\mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ induces a heart-exact functor $\mathcal{C} \to \mathcal{D}$. This concludes. $\square$

**Remark 10.12** Differently stated, the above Corollary says that an exact functor $\mathcal{C} \to \mathcal{D}$ between bounded heart categories is heart-exact if and only if it maps $\mathcal{C}^\heartsuit$ to $\mathcal{D}^\heartsuit$.

Only one question remains: what is the essential image of the heart functor of heart categories. This is the content of joint work with Christoph Winges, which has appeared in [SW25] and which we now present:

**Proposition 10.13** Let $\mathcal{E}$ be an exact category and denote $\mathcal{E}^\oplus$ the underlying additive category, then the canonical map $p : \mathrm{Stab}(\mathcal{E}^\oplus) \to \mathrm{Stab}(\mathcal{E})$ is a Verdier projection.

**Proof.** The functor $\mathrm{Ind}(p)$ identifies with

$$\mathrm{Fun}^\oplus(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp})$$

which is left adjoint to the precomposition. Since this adjoint is fully-faithful, $\mathrm{Ind}(p)$ is a localization. Moreover, remark that the kernel of $\mathrm{Ind}(p)$ is compactly-generated; indeed, it is generated by those additive functors $F : \mathcal{E}^{\mathrm{op}} \to \mathrm{Sp}$ of the form

$$\mathrm{cofib}(\mathrm{cofib}(j(X) \hookrightarrow j(Y)) \to j(Z))$$

for a short-exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$, with $j$ the Yoneda embedding of $\mathcal{E}$, and this colimit is finite, hence preserves compactness. This implies that $p$ is a Karoubi projection, and it is a Verdier projection if and only if it is essentially surjective by [CDH+23b, Corollary A.3.9]. This follows from the proof of [Kle23]. $\square$

In particular, since one can build heart (and in fact weight) structures for stable envelope of additive categories by work of Bondarko (see also [Sau23b, Proposition 2.13]), it suffices to see:

**Proposition 10.14 — S.-Winges.** Let $\mathcal{E}$ be an exact category. There is a bounded heart structure on $\mathrm{Stab}(\mathcal{E})$ such that the functor $p : \mathrm{Stab}(\mathcal{E}^{\oplus}) \to \mathrm{Stab}(\mathcal{E})$ is heart-exact.

**Proof.** Write $\mathrm{Stab}(\mathcal{E})_{\geq 0}$ for the image of $\mathrm{Stab}(\mathcal{E}^{\oplus})_{\geq 0}$ under $p$. Then, $\mathrm{Stab}(\mathcal{E})_{\geq 0}$ is closed under extensions and finite colimits in $\mathrm{Stab}(\mathcal{E})$ because this already holds before applying $p$. Recall that Stab commutes with taking opposite. Letting $\mathrm{Stab}(\mathcal{E})_{\leq 0}$ be the opposite of the image of $(\mathrm{Stab}(\mathcal{E}^{\mathrm{op}})_{\geq 0})$ in $\mathrm{Stab}(\mathcal{E}^{\mathrm{op}})$, one sees that the above arguments dualize to show that $\mathrm{Stab}(\mathcal{E})_{\leq 0}$ is closed under finite limits and extensions.

It also follows that there exists the wanted decompositions since they exist in $\mathrm{Stab}(\mathcal{E}^{\oplus})$ and $p$ is an essentially surjective, exact functor. The boundedness is also obvious. $\square$

Recall that an additive category $\mathcal{A}$ is said to be *weakly-idempotent complete* if every retract diagram $i : A \to X$, $r : X \to A$ and $r \circ i = \mathrm{id}_A$ comes with an equivalence $X \simeq A \oplus B$ such that $r$ is the projection onto $A$. Equivalently, $\mathcal{A}$ has fibers of split projections. The following is proven in more details in Proposition 5.3 of [SW25].

**Theorem 10.15 — S.-Winges.** Suppose $\mathcal{E}$ is weakly-idempotent complete. Then, $\mathcal{E}$ identifies as the heart of the heart structure on $\mathrm{Stab}(\mathcal{E})$

**Proof.** If $\mathcal{E}$ is endowed with a split-exact structure, then this is the content [Sau23b, Proposition 2.13] (see also Sosnilo's proof [Sos19, Corollary 3.4]) and we will not reproduce the proof here. In particular, this applies to the weight structure on $\mathrm{Stab}(\mathcal{E}^{\oplus})$. We now seek to extend this result to non-split exact $\mathcal{E}$.

Since $\mathcal{E}$ is weakly-idempotent complete, it is closed under fibers of projections which lie in $\mathrm{Stab}(\mathcal{E})_{\geq 0}$ (see [SW25, Theorem 3.13], but the hard direction follows essentially from the proof of [Kle23, Proposition 4.25]). Moreover, the heart structure on $\mathrm{Stab}(\mathcal{E})$ is built through the projection of Proposition 10.13:

$$p : \mathrm{Stab}(\mathcal{E}^{\oplus}) \longrightarrow \mathrm{Stab}(\mathcal{E})$$

Since $p$ is a heart-exact functor, it carries the heart to the heart. Any object in $\mathrm{Stab}(\mathcal{E})^{\heartsuit}$ is witnessed as an object of $\mathrm{Stab}(\mathcal{E})_{\geq 0}$ by some resolution (and dually for $\mathrm{Stab}(\mathcal{E})_{\leq 0}$) but such a resolution must come from a resolution in $\mathrm{Stab}(\mathcal{E}^{\oplus})_{\geq 0}$ by construction. In particular, $p$ is essentially surjective on the heart. But by naturality, the composite

$$\mathcal{E}^{\oplus} \longrightarrow \mathrm{Stab}(\mathcal{E}^{\oplus}) \longrightarrow \mathrm{Stab}(\mathcal{E})$$

factors through $\mathcal{E}$. Hence $\mathrm{Stab}(\mathcal{E})^{\heartsuit}$ identifies with $\mathcal{E}$. $\square$

## 10.2 A resolution theorem for K-theory

We prove the resolution theorem for K-theory using the Q-construction; in particular, this part of the argument depends on more than just the universal property for K-theory. We refer to [Bar13] for a self-contained reference on the Q-construction.

Another proof of the resolution theorem exists, due to Staffeldt in [Sta89], which works using the the $S_\bullet$ construction which can also be adapted to the higher categorical world[12]

**Theorem 10.16 — Quillen's resolution theorem.** Let $i : \mathcal{A} \to \mathcal{C}$ be a resolving functor. Then, $i$ induces an equivalence of spaces $|Q(\mathcal{A})| \xrightarrow{\simeq} |Q(\mathcal{C})|$ so consequently an equivalence on the level

[12] And has been in the lecture notes by Christoph Winges.

of K-theory.

**Proof.** Denote $\mathcal{B}$ the full subcategory of $Q(\mathcal{C})$ spanned by the image of $Q(\mathcal{A})$. We have a factorization

$$Q(\mathcal{A}) \xrightarrow{\ g\ } \mathcal{B} \xrightarrow{\ f\ } Q(\mathcal{C})$$

and we will show that both maps induce equivalences on the geometric realizations by showing that $g^{\mathrm{op}}$ and $f$ are cofinal. For both proofs, we use the higher categorical version of Quillen's Theorem A (see [Lur08, Theorem 4.1.3.1]), and are reduced to show some categories are weakly contractible.

We first show $f$ is a weak equivalence. Let $X \in Q(\mathcal{C})$, it suffices to show that $\mathcal{M} := \mathcal{B} \times_{Q(\mathcal{C})} Q(\mathcal{C})_{X/}$ is contractible; by construction, $\mathcal{M}$ is a category whose objects are spans $X \twoheadleftarrow Z \hookrightarrow A$ with $A \in \mathcal{A}$. Consider the wide subcategory $Q^{proj}(\mathcal{C})$ of $Q(\mathcal{C})$ composed of spans $X \twoheadleftarrow Z \hookrightarrow Y$ where the map $Z \to Y$ is an equivalence; this category is equivalent to the wide subcategory $(\mathcal{C}^{proj})^{\mathrm{op}}$ of $\mathcal{C}^{\mathrm{op}}$ whose arrows are exact projections.

Denote $\mathcal{M}^{proj}$ the subcategory of $\mathcal{M}$ given by $\mathcal{B} \times_{Q(\mathcal{C})} Q^{proj}(\mathcal{C})_{X/}$; this a full subcategory of $\mathcal{M}$ since it is equivalently the pullback along the projection from $\mathcal{M}$ of $Q^{proj}(\mathcal{C})_{X/} \to Q(\mathcal{C})_{X/}$ which is easily checked to be fully-faithful. Its objects are equivalently exact projections $A \twoheadrightarrow X$ with $A \in \mathcal{A}$ and a map from $A \twoheadrightarrow X$ to $A' \twoheadrightarrow X$ is the datum of an exact projection $A' \twoheadrightarrow A$ and a homotopy which makes the obvious triangle commute (note the order reversal).

By [HHLN20, Proposition 4.9], the collection of spans where the right hand map an equivalence (purely forward pointing spans in the parlance of *loc. cit.*) and spans where the left hand map is the identity (purely backward pointing spans) forms an orthogonal factorization system on $Q(\mathcal{C})$ (see [Lur08, Definition 5.2.8.8]). It follows from Lemma 5.2.8.19 of *loc. cit.* that the inclusion $Q^{proj}(\mathcal{C})_{X/} \to Q(\mathcal{C})_{X/}$ admits a right adjoint given on objects as follows:

$$(X \twoheadleftarrow Y \hookrightarrow Z) \longmapsto (X \twoheadleftarrow Y \xlongequal{\mathrm{id}_Y} Y)$$

For any span $X \twoheadleftarrow Z \hookrightarrow Y$ with $Y \in \mathcal{A}$, we have $Z \in \mathcal{A}$ as well hence the above descends to a functor $\mathcal{M} \to \mathcal{M}^{proj}$. One readily checks that it provides a right adjoint to the inclusion $\mathcal{M}^{proj} \to \mathcal{M}$ hence the latter is a homotopy equivalence. Thus we are reduced to showing that $\mathcal{M}^{proj}$ is weakly contractible.

By (ii), the category $\mathcal{M}^{proj}$ is nonempty since there exists $B \twoheadrightarrow X$ with $B \in \mathcal{A}$. Moreover, given two projections $Y \twoheadrightarrow X$ and $Y' \twoheadrightarrow X$, then closure under extensions of $\mathcal{A}$ implies that $Y \times_X Y'$ is an object of $\mathcal{A}$ since it fits in the following exact sequence of $\mathcal{C}$:

$$Z \longrightarrow Y \times_X Y' \longrightarrow Y$$

where we have taken $Z$ to be the object fitting in an exact sequence $Z \hookrightarrow Y \twoheadrightarrow X$, which by (i) implies that $Z \in \mathcal{A}$. In particular, by fixing some object $p : A_0 \twoheadrightarrow X$ in $\mathcal{M}^{proj}$, and denoting $P$ the functor sending $A \twoheadrightarrow X$ to $A \times_X A_0 \twoheadrightarrow X$, one obtains two natural transformations $\mathrm{id} \implies P$ and $\mathrm{cst}_p \implies P$ which together imply that $\mathcal{M}^{proj}$ is weakly contractible.

In order to show that $g$ is a weak equivalence, we let $X \in \mathcal{B}$ and working dually, we show that $\mathcal{N} := Q(\mathcal{A}) \times_{\mathcal{B}} \mathcal{B}_{/X}$ is weakly contractible; this time, $\mathcal{N}$ is a category whose objects are spans $Y \twoheadleftarrow Z \hookrightarrow X$ (mind the order) and all three $X, Y, Z \in \mathcal{A}$ though the maps of the span are only exact inclusions/projections in the exact structure on $\mathcal{C}$. However, note that a map in $\mathcal{N}$ from $(Y \twoheadleftarrow Z \hookrightarrow X)$ to $(Y' \twoheadleftarrow Z' \hookrightarrow X)$ are given by diagrams of the following shape:

$$\begin{tikzcd}
Y & T \arrow[l, two heads, "\in\mathcal{A}"'] \arrow[d, hook, "\in\mathcal{A}"] & Z \arrow[l, two heads] \arrow[d, hook] \\
 & Y' & Z' \arrow[l, two heads] \arrow[d, hook] \\
 & & X
\end{tikzcd}$$

where every object is in $\mathcal{A}$, the maps indicated $\in \mathcal{A}$ are exact injections/projections in the exact structure of $\mathcal{A}$ and the square is cartesian in $\mathcal{C}$; in particular, it follows from this last condition that $Z \hookrightarrow Z'$ is already an exact inclusion in the exact structure of $\mathcal{A}$.

Dually to what have done before, we can consider $Q^{inj}(\mathcal{C})$ the wide subcategory of $Q(\mathcal{C})$ whose arrows are spans $Y \twoheadleftarrow Z \hookrightarrow X$ with $Z \to Y$ an equivalence, and $\mathcal{B}^{inj}$ the full subcategory of $Q^{inj}(\mathcal{C})$ spanned by $Q^{inj}(\mathcal{A})$. Let $\mathcal{N}^{inj}$ be the subcategory of $\mathcal{N}$ given by $Q(\mathcal{A}) \times_{\mathcal{B}} \mathcal{B}^{inj}_{/X}$; again, $\mathcal{N}^{inj}$ is a full subcategory of $\mathcal{N}$. It has equivalently objects exact inclusions $Z \hookrightarrow X$ and morphisms given by exact inclusions $Z \overset{\in\mathcal{A}}{\hookrightarrow} Z'$ featuring in an exact sequence of $\mathcal{A}$ as well as a homotopy which makes the obvious diagram commute (note there is no order reversal here).

Finally, remark that the left adjoint to the inclusion $Q^{inj}(\mathcal{C})_{/X} \to Q(\mathcal{C})_{/X}$ which only keeps non-trivial the exact-inclusion, sends $\mathcal{B}_{X/}$ to $\mathcal{B}^{inj}_{X/}$ and $Q(\mathcal{A})_{/X}$ to $Q^{inj}(\mathcal{A})_{/X}$, so that it descends to $\mathcal{N}$ by virtue of (i). Hence, $\mathcal{N}^{inj} \to \mathcal{N}$ is a homotopy equivalence. To conclude, remark that $\mathcal{N}^{inj}$ has an initial object, given by the span $0 \twoheadleftarrow 0 \hookrightarrow X$. □

**Remark 10.17** The situation sometimes calls for a dual version of the above. Indeed, recall that $\mathrm{K}(\mathcal{C}) \simeq \mathrm{K}(\mathcal{C}^{\mathrm{op}})$ so that the Lemma also applies if $i$ is op-resolving, i.e. if the following two conditions are met:

(i') For every exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$ with $Y \in \mathcal{A}$, we have $Z \in \mathcal{A}$

(ii) For every $X \in \mathcal{C}$, there is an exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$ with $Y \in \mathcal{A}$.

The above result provides a counterpart to Theorem 3 of [Qui73]. Theorem 10.10 shows if $\mathcal{C}$ has a heart structure, then the map $\mathcal{C}^{\heartsuit} \to \mathcal{C}$ factors as a transfinite composition of resolving and op-resolving functors. The theorem of the heart for heart structures immediately follows:

**Theorem 10.18 — Theorem of the heart.** Let $\mathcal{C}$ be a bounded heart structure and denote $\mathcal{C}^{\heartsuit}$ its heart. Then, the map

$$\mathrm{K}(\mathcal{C}^{\heartsuit}) \xrightarrow{\simeq} \mathrm{K}(\mathcal{C})$$

is an equivalence, where on the left hand side, we have taken K-theory of the exact category $\mathcal{C}^{\heartsuit}$.

**Proof.** This follows from Theorem 10.10 combined with Theorem 10.16 using that K preserves filtered colimits. □

If $\mathcal{C}$ admits a heart structure, there is a subclass of bimodules $M$ which are suitably adapted to the weight structure. This is the following definition:

**Definition 10.19** Let $\mathcal{C}$ be a heart category. A $\mathcal{C}$-bimodule $M$ is said to be a *weighted bimodule* if for every $X \in \mathcal{C}_{\leq 0}$ and $Y \in \mathcal{C}_{\geq 0}$, the following mapping spectra

$$\mathrm{map}_{\mathrm{Ind}\,\mathcal{C}}(X, M(Y))$$

is connective.

■ **Example 10.20** Let $R$ be a connective ring spectra. Recall $\mathrm{Perf}(R)$ admits a weight structure, as it is the stable envelope of the additive $\mathrm{Proj}(R)$. A $R$-bimodule is weighted if and only for every connective $N$ and coconnective $P$, the spectrum

$$\mathrm{map}_{\mathrm{Mod}_R}(P, M \otimes_R N)$$

is connective. Applied to $N = R$, one checks that $M$ is necessarily connective, and this is also sufficient by the combination of the following two facts: whenever $P$ is coconnective and $Q$ is connective, $\mathrm{map}(P, Q)$ is itself connective and $M \otimes_R -$ preserves connectivity when $M$ is connective. Hence, weighted $R$-bimodules are exactly connective $R$-bimodules. ■

In the situation where $\mathcal{C}$ admits a weight structure and $M$ is a weighted bimodule, $\mathrm{Lace}(\mathcal{C}, M)$ inherits a heart structure whose heart is $\mathrm{Lace}(\mathcal{C}^{\heartsuit}, M)$, the full subcategory of pairs $(X, f)$ with $X \in \mathcal{C}^{\heartsuit}$. This is not quite a weight structure, because the mapping spectra in $\mathrm{Lace}(\mathcal{C}^{\heartsuit}, M)$ need only be $(-1)$-connective; in fact, it is this fact that started the investigation which led to this article, and the introduction of heart structures.

**Lemma 10.21** Let $\mathcal{C}$ be a heart category and $M$ a weighted bimodule. Then, $\mathrm{Lace}(\mathcal{C}, M)$ admits a heart structure given by the full subcategories $\mathrm{Lace}(\mathcal{C}_{\geq 0}, M)$ and $\mathrm{Lace}(\mathcal{C}_{\leq 0}, M)$ fibered respectively over $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$

**Proof.** Clearly, $\mathrm{Lace}(\mathcal{C}_{\geq 0}, M)$ and $\mathrm{Lace}(\mathcal{C}_{\leq 0}, M)$ are closed under retracts and extensions in $\mathrm{Lace}(\mathcal{C}, M)$, the former under pushouts and the latter under pullbacks.

If $(Z, h : Z \to M(Z))$ is an object $\mathrm{Lace}(\mathcal{C}, M)$, then there exists a weight decomposition $X \to Z \to \Sigma Y$, i.e. an exact sequence of $\mathcal{C}$ with $X \in \mathcal{C}_{\leq 0}$ and $Y \in \mathcal{C}_{\geq 0}$. We have two diagrams

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "p"] & M(Z) \arrow[d, "M(p)"] \\ \Sigma Y \arrow[r, dotted] & \Sigma M(Y) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dotted] \arrow[d, "i"] & M(X) \arrow[d, "M(i)"] \\ Z \arrow[r, "h"] & M(Z) \end{tikzcd}$$

which it suffices to fill via the dotted arrows to get (iii) of Definition 10.7. For the left hand side, we ought to show that the image of the map

$$\mathrm{map}(\Sigma Y, \Sigma M(Y)) \xrightarrow{p^*} \mathrm{map}(Z, \Sigma M(Y))$$

contains $M(p) \circ h$. The cofiber of this map is exactly $\mathrm{map}(X, \Sigma M(Y))$ which is 1-connective. Hence the above map is essentially surjective on $\pi_0$ providing the wanted dotted arrow. A dual argument deals with the other square. $\square$

**Remark 10.22** In the situation of the previous lemma, if the heart structure on $\mathcal{C}$ is in fact a weight structure — equivalently, the exact structure $\mathcal{C}^{\heartsuit}$ is split-exact — then $\mathrm{Lace}(\mathcal{C}, M)$ verifies the following weaker version of axiom (ii) of a weight structure:

If $(X, f : X \to M(X))$ and $(Y, g : Y \to M(Y))$ with $X \in \mathcal{C}_{\leq 0}$ and $Y \in \mathcal{C}_{\geq 0}$, then

$$\mathrm{Map}_{\mathrm{Lace}(\mathcal{C},M)}((X, f), (Y, g))$$

is $(-1)$-connective.

This can be checked using the explicit formula for mapping spaces of lax-equalizers [NS17, Proposition II.1.5.(ii)].

Applying Theorem 10.18 to this situation, we have the following:

**Corollary 10.23** Let $\mathcal{C}$ be a bounded heart structure and $M$ a weighted bimodule. Denote $\mathrm{Lace}(\mathcal{C}^{\heartsuit}, M)$ the full subcategory of $\mathrm{Lace}(\mathcal{C}, M)$ fibered over $\mathcal{C}^{\heartsuit}$. Then, there is an equivalence:

$$\mathrm{K}(\mathrm{Lace}(\mathcal{C}^{\heartsuit}, M)) \xrightarrow{\simeq} \mathrm{K}(\mathrm{Lace}(\mathcal{C}, M))$$

In particular, when the heart structure on $\mathcal{C}$ is a weight structure, the Yoneda embedding is one such weighted $\mathcal{C}$-bimodule. Hence, the previous Corollary specifies to the following:

**Theorem 10.24 — Theorem of the heart for $\mathrm{K}^{\mathrm{End}}$.** Let $\mathcal{C}$ be a stable category equipped with a bounded weight structure. Then, there is an equivalence

$$\mathrm{K}(\mathrm{End}(\mathcal{C}^{\heartsuit})) \xrightarrow{\simeq} \mathrm{K}(\mathrm{End}(\mathcal{C}))$$

We finish this section by leveraging Corollary 10.23 to show that if $\mathcal{C}$ has a bounded heart structure, then $\mathrm{K}(\mathrm{Lace}(\mathcal{C}, M))$ commutes with sifted colimits taking values in weighted bimodules $M$. This kind of argument plays a key role in the proof of the Dundas-Goodwillie-McCarthy theorem (see [DGM13] for an account of this theorem in the setting of connective ring spectra), and will be used in a future article to provide a new proof of this theorem in the context of stable categories, without referring to ring spectra.

**Proposition 10.25** Let $\mathcal{C}$ be a bounded heart category, then when restricted to the full subcategory spanned by weighted bimodules, the functor $\mathrm{K}(\mathrm{Lace}(\mathcal{C}, -))$ preserves sifted colimits.

**Proof.** Combining Lemma 10.21 and Theorem 10.18, we see that in the case where the sifted colimit has values in weighted bimodules, it suffices to prove that $\mathrm{K}(\mathrm{Lace}(\mathcal{C}^\heartsuit, M))$ commutes with sifted colimits. Note also that weighted bimodules are closed under colimits in $\mathrm{Bimod}(\mathcal{C})$ so this colimit is also computed therein.

Recall that $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}$ preserves sifted colimits by [Lur17a, 1.4.3.9] hence it suffices to show the result for the space-valued delooped K-theory. Instead of the Q-construction, we will use the description of this space as the geometric realization of the core of the $S_\bullet$-construction, thanks to [Bar13, Theorem 3.10]. Since geometric realizations preserve colimits, it suffices to show that each $\iota\, S_n \,\mathrm{Lace}(\mathcal{C}^\heartsuit, -)$ preserves sifted colimits of weighted bimodules.

As a (necessary) warm-up, let us treat the first non-trivial case, $n = 1$, where we ought to show that $\iota\,\mathrm{Lace}(\mathcal{C}^\heartsuit, M)$ preserves sifted colimits in weighted $M$. We claim that for a fixed $X \in \mathcal{C}^\heartsuit$, the space $\mathrm{Map}_{\mathrm{Ind}\,\mathcal{C}}(X, M(X))$ commutes with sifted colimits in weighted $M$. Indeed, this is clear at the level of the mapping spectra because $X$ is compact in $\mathrm{Ind}\,\mathcal{C}$, and as we have already used $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}$ preserves sifted colimits, so when $M$ is weighted, we have the first claim.

But since $\iota\,\mathrm{Lace}(\mathcal{C}^\heartsuit, M)$ is fibered over such spaces by 3.13, this extends to the whole space. Indeed, the unstraightening of a functor is its lax-colimit and lax-colimits commute with sifted colimits (combine the formula of [GHN17, Definition 2.9] and the fact that sifted colimits of spaces commute to products).

Recall that $S_n(\mathrm{Lace}(\mathcal{C}^\heartsuit, M))$ is the category of the following diagrams:

$$\begin{tikzcd}
0 \arrow[r, hook] & (X_{1,1}, f_{1,1}) \arrow[r, hook] \arrow[d, two heads] & (X_{1,2}, f_{1,2}) \arrow[r, hook] \arrow[d, two heads] & \dots \arrow[r, hook] \arrow[d, two heads] & (X_{1,n}, f_{1,n}) \arrow[d, two heads] \\
 & 0 \arrow[r, hook] & (X_{2,2}, f_{2,2}) \arrow[r, hook] \arrow[d, two heads] & \dots \arrow[r, hook] \arrow[d, two heads] & (X_{2,n}, f_{2,n}) \arrow[d, two heads] \\
 & & \dots \arrow[r, hook] & \dots \arrow[r, hook] \arrow[d, two heads] & \dots \arrow[d, two heads] \\
 & & & 0 \arrow[r, hook] & (X_{n,n}, f_{n,n}) \arrow[d, two heads] \\
 & & & & 0
\end{tikzcd}$$

where $(X_{i,j}, f_{i,j}) \in \mathrm{Lace}(\mathcal{C}^\heartsuit, M)$ and every square is exact. Of course, such a diagram is fully determined by its first row but remark that not all first rows need to induce a diagram where every object lies in $\mathrm{Lace}(\mathcal{C}^\heartsuit, M)$; in fact, this is already the case for $\mathcal{C}^\heartsuit$ in $\mathcal{C}$.

Denote $S_n(M)$ the induced bimodule on $S_n(\mathcal{C})$, obtained by applying $M$ to the diagrams of the above shape, then the above category is none other than $\mathrm{Lace}(S_n(\mathcal{C}^\heartsuit), S_n(M))$, viewed as a full subcategory of $\mathrm{Lace}(S_n(\mathcal{C}), S_n(M))$. The arguments of the case $n = 1$ will conclude, provided we can show that $S_n(M)$ is still a weighted bimodule, i.e. for every object $X := (X_{i,j}) \in \mathrm{Lace}(S_n(\mathcal{C}^\heartsuit), S_n(M))$ the mapping spectra $\mathrm{map}(X, S_n(M)(X))$ is connective. Such a mapping spectra is given by iterated pullbacks (because in $\mathcal{C}$, the above diagrams are exactly equivalent to the datum of the first row), hence we have to show that

$$\mathrm{map}(X_{1,1}, M(X_{1,1})) \times_{\mathrm{map}(X_{1,1}, M(X_{1,2}))} \dots \times_{\mathrm{map}(X_{1,n-1}, M(X_{1,n}))} \mathrm{map}(X_{1,n}, M(X_{1,n}))$$

is connective, where all mapping spectra are taken in $\mathrm{Ind}\,\mathcal{C}$. All of the terms appearing in the above are connective, so by induction, it suffices to show that every $\mathrm{map}(X_{1,i+1}, M(X_{1,i+1})) \to \mathrm{map}(X_{1,i}, M(X_{1,i+1}))$ is surjective on $\pi_0$. But $X_{1,i} \hookrightarrow X_{1,i+1}$ has its cofiber in $\mathcal{C}^\heartsuit$ and $M$ is exact and weighted, which concludes. ☐

## 10.3 A resolution theorem for THH

Our goal is to show that THH also satisfies the resolution theorem (and thus a theorem of the heart) in the same setting that made it hold for K-theory in the previous section. For this, we will first need to extend our definition of THH to the laced-exact world, i.e. for bimodules of exact categories.

Let us denote $\mathrm{Bimod}(\mathbf{Exact}_\infty)$ the bicartesian unstraightening of the functor $\mathbf{Exact}_\infty \to \mathbf{CAT}_\infty$ sending $\mathcal{E}$ to $\mathrm{Fun}^{\mathrm{BiEx}}(\mathcal{E}^{\mathrm{op}} \times \mathcal{E}, \mathrm{Sp})$. The category $\mathrm{Bimod}(\mathbf{Exact}_\infty)$ contains $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$ as a full subcategory, which is exactly its pullback along the fully-faithful $\mathbf{Cat}^{\mathrm{Ex}} \to \mathbf{Exact}_\infty$. Note that this has simply thickened $\mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$, i.e. duplicated some fibers:

**Lemma 10.26** Let $\mathcal{E}$ be an exact category and denote $i : \mathcal{E} \to \mathrm{Stab}(\mathcal{E})$ the canonical functor, then restriction along $i^{\mathrm{op}} \times i$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{BiEx}}(\mathrm{Stab}(\mathcal{E})^{\mathrm{op}} \times \mathrm{Stab}(\mathcal{E}), \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{BiEx}}(\mathcal{E}^{\mathrm{op}} \times \mathcal{E}, \mathrm{Sp})$$

In particular, $\mathrm{Bimod}(\mathrm{Stab}(\mathcal{E}))$ is generated under filtered colimits by functors of the form $\mathrm{map}(X, -) \otimes \mathrm{map}(-, Y)$ where $X, Y \in \mathcal{E}$ and the mapping spectra are taken in $\mathrm{Stab}(\mathcal{E})$.

**Proof.** Remark that $\mathrm{Stab}(\mathcal{E}^{\mathrm{op}}) \simeq \mathrm{Stab}(\mathcal{E})^{\mathrm{op}}$ because they have the same universal property:

$$\begin{aligned}\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{E}^{\mathrm{op}}, \mathcal{C}) &\simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{E}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \\ &\simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Stab}(\mathcal{E}), \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \\ &\simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Stab}(\mathcal{E})^{\mathrm{op}}, \mathcal{C})\end{aligned}$$

if $\mathcal{C}$ is stable. It follows that the stable category $\mathcal{P}_\Sigma(\mathcal{E}) := \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp})$ is the Ind-completion of $\mathrm{Stab}(\mathcal{E})$. Then, we have

$$\mathrm{Fun}^{\mathrm{BiEx}}(\mathcal{E}^{\mathrm{op}} \times \mathcal{E}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathcal{E}, \mathcal{P}_\Sigma(\mathcal{E})) \simeq \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Stab}(\mathcal{E}), \mathrm{Ind}\,\mathrm{Stab}(\mathcal{E}))$$

where the right hand side is exactly $\mathrm{Stab}(\mathcal{E})$-bimodules as wanted. $\square$

**Definition 10.27** Let $\mathcal{E}$ be an exact category and $M : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathrm{Sp}$ be a bi-exact functor, then, we let

$$\mathrm{THH}(\mathcal{E}, M) := \int^{X \in \mathcal{E}} M(X, X)$$

This extends THH to a functor $\mathrm{Bimod}(\mathbf{Exact}_\infty) \to \mathrm{Sp}$ which is still fiberwise-exact and trace-invariant, for suitable generalizations of these definitions.

It follows from Lemma 10.26 that the behavior of a fiberwise filtered-colimit preserving invariant is fully-determined by its values on bimodules who split as a tensor of a left module and a right module. We now investigate the behaviour of THH on such objects. The result we prove is a very general statement about coends, in the following situation.

Consider a subcategory $\mathfrak{C}$ of $\mathbf{Cat}_\infty$ and a presentably symmetric monoidal $\mathcal{T} \in \mathcal{C}$, such that if $f : \mathcal{C} \to \mathcal{D}$ is a map in $\mathcal{C}$, there is a natural adjunction

$$\mathrm{Fun}_{\mathfrak{C}}(\mathcal{D}, \mathcal{T}) \underset{f^*}{\overset{f_!}{\leftrightarrows}} \mathrm{Fun}_{\mathfrak{C}}(\mathcal{C}, \mathcal{T})$$

where $f^*$ denotes the precomposition functor and $f_!$ its left adjoint (which need not be the left Kan extension since we are working internally to $\mathfrak{C}$). Then, we have:

**Proposition 10.28** Let $M : \mathcal{C}^{\mathrm{op}} \to \mathcal{T}$ and $N : \mathcal{D} \to \mathcal{T}$ be maps of $\mathfrak{C}$ and let $f : \mathcal{C} \to \mathcal{D}$ be a map of of $\mathfrak{C}$. Then, $f$ induces an equivalence in $\mathcal{T}$:

$$\alpha(f) \colon \int^{\mathcal{C}} M \otimes f^*(N) \xrightarrow{\simeq} \int^{\mathcal{D}} f_!(M) \otimes N$$

where the tensor is the symmetric monoidal structure of $\mathcal{E}$, taken pointwise on functors.

**Proof.** Let us first explain what is the induced map: there is natural transformation $\mathrm{id} \implies f_! f^*$

which induces a natural transformation of functors, and finally a map of coend

$$\int^{\mathcal{C}} M \otimes f^* N \longrightarrow \int^{\mathcal{C}} f_! f^*(M) \otimes f^* N$$

Moreover, there is also a map

$$\int^{\mathcal{C}} f_! f^*(M) \otimes f^* N \longrightarrow \int^{\mathcal{D}} f_!(M) \otimes N$$

induced by restricting the colimit along the induced $\mathrm{TwAr}(f)$. The composite is the wanted map.

Fix $Z \in \mathcal{T}$, we will show that the wanted map is a $\mathcal{T}$-equivalence by showing that $\alpha(f)^*$ is an equivalence:

$$\alpha(f)^* \colon \mathrm{Map}(\int^{\mathcal{D}} f_!(M) \otimes N, Z) \longrightarrow \mathrm{Map}(\int^{\mathcal{C}} M \otimes f^* N, Z)$$

The contravariant side of Map sends colimits to limits, and so we are equivalently trying to show that:

$$\int_{\mathcal{D}} \mathrm{Map}(f_!(M) \otimes N, Z) \longrightarrow \int_{\mathcal{C}} \mathrm{Map}(M \otimes f^* N, Z)$$

is an equivalence. Using that $\mathcal{T}$ is presentably symmetric monoidal and denoting $\underline{\mathrm{Map}}$ the internal mapping functor, we can rewrite the above mapping spaces as follows:

$$\int_{\mathcal{D}} \mathrm{Map}(f_!(M), \underline{\mathrm{Map}}(N, Z)) \longrightarrow \int_{\mathcal{C}} \mathrm{Map}(M, f^* \underline{\mathrm{Map}}(N, Z))$$

where $\underline{\mathrm{Map}}(N, Z)$ is the functor sending $X \in \mathcal{D}$ to $\underline{\mathrm{Map}}(N(X), Z) \in \mathcal{T}$ (so that the commutation with $f^*$ on the right hand side is legitimate).

Now by definition of the space of natural transformations, we can further rewrite both terms as:

$$\mathrm{Nat}(f_!(M), \underline{\mathrm{Map}}(N, Z)) \longrightarrow \mathrm{Nat}(M, f^* \underline{\mathrm{Map}}(N, Z))$$

and that by construction, the map we consider is the one of the adjunction between $f_!$ and $f^*$, so it is an equivalence as wanted. $\square$

Combining Lemma 10.26 and Proposition 10.28, we get in particular:

**Corollary 10.29** Let $\mathcal{C}, \mathcal{D}$ be exact categories and $M : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Sp}$ be a $(\mathcal{C}, \mathcal{D})$-bimodule, then any exact $f : \mathcal{C} \to \mathcal{D}$ induces an equivalence:

$$\mathrm{THH}(\mathcal{C}, (\mathrm{id} \times f)_* M) \xrightarrow{\simeq} \mathrm{THH}(\mathcal{D}, (f^{\mathrm{op}} \times \mathrm{id})_! M)$$

where $f_!$ denotes left Kan extension followed by taking the exact approximation.

Note that we can also describe $(f^{\mathrm{op}} \times \mathrm{id})_! M$ as the $\mathrm{Stab}(\mathcal{D})$-bimodule left Kan extended from $\mathrm{Stab}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Stab}(\mathcal{D})$ along $\mathrm{Stab}(f)^{\mathrm{op}} \times \mathrm{id}$, and this description need not involve further exact approximation since this is automatic for exact functors between stable categories.

**Theorem 10.30** Let $\mathcal{E}$ be an exact category, $M \in \mathrm{Bimod}(\mathcal{E})$ and denote $i : \mathcal{E} \to \mathrm{Stab}(\mathcal{E})$ the canonical functor. Then, $i$ induces a map in $\mathrm{Bimod}(\mathbf{Exact}_\infty)$ such that

$$\mathrm{THH}(\mathcal{E}, M) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{Stab}(\mathcal{E}), \mathrm{Stab}(M))$$

is an equivalence.

**Proof.** Let $M$ be an exact functor $\mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathrm{Sp}$ and denote $\mathrm{Stab}(M)$ its unique extension to an exact functor $\mathrm{Stab}(\mathcal{E})^{\mathrm{op}} \times \mathrm{Stab}(\mathcal{E}) \to \mathrm{Sp}$. Write $i : \mathcal{E} \to \mathrm{Stab}(\mathcal{E})$ for the exact inclusion and consider $\widehat{M} := \mathrm{Stab}(M)(i(-), -)$ which is a $(\mathcal{E}, \mathrm{Stab}(\mathcal{E}))$-bimodule[13]. By the above proposition, it holds that $i$ induces an equivalence

$$\mathrm{THH}(\mathcal{E}, (\mathrm{id} \times i)_* \widehat{M}) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{Stab}(\mathcal{E}), (i^{\mathrm{op}} \times \mathrm{id})_! \widehat{M})$$

[13]Equivalently, $\widehat{M} := (\mathrm{id} \times i)_! M$

Of course, $(\mathrm{id} \times i)_* \widehat{M}$ is just $M$ and on the right hand side, $(i^{\mathrm{op}} \times \mathrm{id})_! \widehat{M}$ is a $\mathrm{Stab}(\mathcal{E})$-bimodule, so it is fully determined by its restriction to $\mathcal{E}^{\mathrm{op}} \times \mathcal{E}$, which is $M$ by fully-faithfulness of $i$. Hence $(i^{\mathrm{op}} \times \mathrm{id})_! \widehat{M} \simeq \mathrm{Stab}(M)$, which concludes. □

In particular, since $\mathrm{Stab}(\mathcal{C}^\heartsuit) \simeq \mathcal{C}$ for any heart category $\mathcal{C}$, we have:

**Corollary 10.31 — Resolution theorem for THH.** If $\mathcal{C}$ is a heart category with heart $\mathcal{C}^\heartsuit$, then the map $\mathrm{THH}(\mathcal{C}^\heartsuit) \to \mathrm{THH}(\mathcal{C})$ is an equivalence.

We can now prove:

**Proposition 10.32** Let $\mathcal{C}$ be a heart category, then $\mathrm{THH}(\mathrm{Lace}(\mathcal{C}, M))$ commutes with sifted colimits in weighted $M$.

**Proof.** Since $\mathcal{C}$ is a heart category and $M$ is weighted, $\mathrm{Lace}(\mathcal{C}, M)$ is a heart category with heart $\mathrm{Lace}(\mathcal{C}^\heartsuit, M)$, hence it suffices to show that $\mathrm{THH}(\mathrm{Lace}(\mathcal{C}^\heartsuit, M))$ has the wanted property. We can realize this spectrum as the following geometric realization:

$$\left| \mathrm{Stab}\,\mathrm{Lace}\left( (\mathrm{Lace}(\mathcal{C}^\heartsuit, M), \mathrm{id})^{([\bullet], *)} \right) \right|$$

where $\mathrm{Stab}\,\mathrm{Lace}$ is a fiberwise-stabilized version of $\Sigma^\infty_+ \mathrm{Lace}^\simeq$, which is given pointwise by the colimit

$$\mathrm{Stab}\,\mathrm{Lace}(\mathcal{C}, M) := \operatorname*{colim}_{X \in \iota\, \mathcal{C}} M(X, X)$$

It suffices to prove that each simplicial layer commutes with sifted colimits in $M$. For $n = 0$, this is $\mathrm{Stab}\,\mathrm{Lace}(\mathrm{Lace}(\mathcal{C}^\heartsuit, M), \mathrm{id})$, which is the colimit

$$\operatorname*{colim}_{(X,f) \in \mathrm{Lace}^\simeq(\mathcal{C}^\heartsuit, M)} \mathrm{map}_{\mathrm{Lace}(\mathcal{C}^\heartsuit, M)}((X, f), (X, f))$$

By [NS17, Proposition II.1.5.(ii)], we have that the above mapping spectra of $\mathrm{Lace}(\mathcal{C}^\heartsuit, M)$ is given by the equalizer

$$E(X, f) := \mathrm{Eq}\left( \mathrm{map}_{\mathcal{C}}(X, X) \underset{M(-) \circ f}{\overset{f \circ (-)}{\rightrightarrows}} \mathrm{map}_{\mathrm{Ind}(\mathcal{C})}(X, M(X)) \right)$$

By Lemma 3.13, $\mathrm{Lace}^\simeq(\mathcal{C}^\heartsuit, M)$ is fibered over $\iota\, \mathcal{C}^\heartsuit$ hence we can split the colimit over $\mathrm{Lace}(\mathcal{C}^\heartsuit, M)$ in two successive colimits, so that

$$\mathrm{Stab}\,\mathrm{Lace}(\mathrm{Lace}(\mathcal{C}^\heartsuit, M), \mathrm{id}) \simeq \operatorname*{colim}_{X \in \mathcal{C}^\heartsuit} \operatorname*{colim}_{f \in \mathrm{Map}(X, M(X))} E(X, f)$$

It suffices to prove for a fixed $X \in \mathcal{C}^\heartsuit$ that $\operatorname{colim}_{f \in \mathrm{Map}(X, M(X))} E(X, f)$ commutes with sifted colimits in $M$. Passing to the colimit over $\mathrm{Map}(X, M(X))$, there is an equalizer diagram of functors $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ (the variable being denoted $M$):

$$\operatorname*{colim}_{f \in \mathrm{Map}(X, M(X))} E(X, f) \longrightarrow \operatorname*{colim}_{f \in \mathrm{Map}(X, M(X))} \mathrm{map}(X, X) \rightrightarrows \operatorname*{colim}_{f \in \mathrm{Map}(X, M(X))} \mathrm{map}(X, M(X))$$

Since the category of functors $\mathrm{T}_{\mathcal{C}}\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{Sp}$ is stable, it suffices to prove that the right side of the diagram has the wanted property and finally, colimits being computed pointwise, that both $\operatorname{colim}_{f \in \mathrm{Map}(X, M(X))} \mathrm{map}(X, X)$ and $\operatorname{colim}_{f \in \mathrm{Map}(X, M(X))} \mathrm{map}(X, M(X))$ commutes with sifted colimits in weighted $M$. These are constant in $f$ so it suffices to show that the indexing space $\mathrm{Map}(X, M(X))$ does: but indeed, $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}$ is sifted-colimit preserving and $\mathrm{map}(X, M(X))$ is connective since $X \in \mathcal{C}^\heartsuit$.

For higher $n \geq 0$, a similar argument applies replacing $\mathcal{C}$ by $(\mathcal{C}^\heartsuit)^{[n]}$, replacing $\mathrm{Map}(X, M(X))$ by $\mathrm{Map}(X_n, M(X_0))$ where $X := (X_i)$ is a chain in $\mathcal{C}^\heartsuit$ and finally, replacing $E(X, f)$ by the mapping spectra in $\mathrm{Lace}(\mathcal{C}^\heartsuit, M)^{([n], *)}$ which is a more complicated limit than an equalizer but nonetheless finite, so that the argument still holds. □

# 11 The local structure of localizing invariants

## 11.1 Weighted bimodules and analycity

Let $\mathcal{C}$ be a heart category, in §10 we have defined a *weighted* $\mathcal{C}$-bimodule to be to a $\mathcal{C}$-bimodule $M : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$ such that the restriction along $(\mathcal{C}^{\heartsuit})^{\mathrm{op}} \times \mathcal{C}^{\heartsuit}$ lands in $\mathrm{Sp}_{\geq 0}$. Denote $\mathrm{Bimod}(\mathcal{C})_{\geq 0}$ the full subcategory of weighted bimodules in $\mathrm{Bimod}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{BiEx}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Sp})$. Note that it is closed under colimits, hence we will be able to use Theorem 9.19 relative to this full subcategory.

We will need the following Lemma:

**Lemma 11.1** Let $(\mathcal{C}, M)$ be a laced category and write $p : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}$ for the canonical map and $R$ for its Ind-right adjoint. Then, there is an equivalence of $\mathcal{C}$-bimodules

$$\mathrm{Ind}(p) \circ R \simeq \mathrm{id} \oplus (M \circ \mathrm{Ind}(p) \circ R)$$

Moreover, if $\mathcal{C}$ has a heart structure such that $M$ is connective, then $M \circ \mathrm{Ind}(p) \circ R$ is also a connective $\mathcal{C}$-bimodule and thus so is the fiber $\mathrm{fib}(\mathrm{Ind}(p) \circ R \to \mathrm{id})$.

**Proof.** Recall that $p$ has a section, $i := \mathrm{Lace}(\mathcal{C}, 0) : \mathcal{C} \to \mathrm{Lace}(\mathcal{C}, M)$ induced by $0 \to M$. The Ind-right adjoint $Q$ of $i$ is easy to describe: it sends a map $f : X \to M(X)$ to its fiber (computed in $\mathrm{Ind}(\mathcal{C})$). In particular, we get a canonical exact sequence $Q \to \mathrm{Ind}(p) \to M \circ \mathrm{Ind}(p)$.

The Ind-right adjoint $R$ of $p$ is a section of $Q$, hence the first map in the sequence induces a section of the canonical $\mathrm{Ind}(p) \circ R \to \mathrm{id}$ after precomposing by $R$. Since $\mathrm{Bimod}(\mathcal{C})$ is stable, id must split as a direct summand of $\mathrm{Ind}(p) \circ R$ and the complementary summand is necessarily $M \circ \mathrm{Ind}(p) \circ R$ using the above exact sequence. This proves the first part.

For the second part, we remark that a bimodule $N : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ is connective if and only if its image is contained in the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ whose restriction to the heart is connective. Indeed, $\mathrm{Ind}(\mathcal{C})$ is generated under colimits by $\mathcal{C}^{\heartsuit}$, and connective bimodule send $\mathcal{C}^{\heartsuit}$ to the aforementioned subcategory, which is closed under colimits in $\mathrm{Ind}(\mathcal{C})$. In particular, this description implies that $\mathrm{Bimod}(\mathcal{C})_{\geq 0}$ is a left-ideal (i.e. post-composing by a connective bimodule any bimodule makes it connective) which suffices by hypothesis. $\square$

Recall that by definition, if $F : \mathbf{Cat}^{\mathrm{Ex}} \to \mathcal{E}$, then we have denoted $F^{\mathrm{cyc}}$ or $F^{\mathrm{red}}$ the fiber $\mathrm{fib}(F(\mathrm{Lace}(\mathcal{C}, M) \to F(\mathcal{C}))$, i.e. the fiberwise-reduction of the left Kan extension of $F$ along the cotangent complex $\mathbf{Cat}^{\mathrm{Ex}} \to \mathrm{T}\mathbf{Cat}^{\mathrm{Ex}}$.

**Proposition 11.2** Let $\mathcal{C}$ be a heart category. The functors $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, -) : \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Sp}$ and $\Sigma\,\mathrm{THH}^{\mathrm{cyc}}(\mathcal{C}, -) : \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Sp}$ are rigid in the sense of Definition 9.8.

**Proof.** We have made the above functors reduced by *fiat*. K-theory always lands in connective spectra and Proposition 10.25 shows $\mathrm{K}^{\mathrm{cyc}}$ preserves sifted colimits of weighted bimodules.

We now turn to $\mathrm{THH}^{\mathrm{cyc}}$. Proposition 10.32 shows preservation of sifted colimits of weighted bimodules so we are left with showing that $\Sigma\,\mathrm{THH}^{\mathrm{cyc}}(\mathcal{C}, M)$ is connective if $M$ is weighted. Recall from the proof of 8.1 that we have:

$$\mathrm{THH}^{\mathrm{lace}}(\mathcal{C}, M) := \mathrm{Eq}\Big(\ \mathrm{THH}(\mathcal{C}, p_M \circ R_M) \rightrightarrows \mathrm{THH}(\mathcal{C}, M \circ p_M \circ R_M)\ \Big)$$

where $p_M : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}$ is the canonical map and $R_M$ denotes its Ind-right adjoint. By Lemma 11.1, we have

$$\mathrm{THH}(\mathcal{C}, p_M \circ R_M) \simeq \mathrm{THH}(\mathcal{C}) \oplus \mathrm{THH}(\mathcal{C}, M \circ p_M \circ R_M)$$

and the composite $M \circ p_M \circ R_M$ is a weighted bimodule. In particular, Theorem 10.30 implies that the second summand $\mathrm{THH}(\mathcal{C}, M \circ p_M \circ R_M)$ is a connective spectra, using the formula of the Definition. Hence, $\mathrm{THH}^{\mathrm{cyc}}$ is given by an equalizer of connective spectra, hence at worst $(-1)$-connective. This shows that $\Sigma\,\mathrm{THH}^{\mathrm{cyc}}(\mathcal{C}, -)$ is rigid as wanted. $\square$

Since $\Sigma\,\mathrm{THH}^{\mathrm{cyc}}(\mathcal{C}, -)$ is rigid, it coincides with the limit of its Taylor tower by Theorem 9.19. But then, this also must hold for $\mathrm{THH}^{\mathrm{cyc}}(\mathcal{C}, -)$ because the shift is invertible and commutes with all possible operations: taking derivatives and passing to the limit of the tower.

**Corollary 11.3** Let $\mathcal{C}$ be a heart category. The functors $\mathrm{K}^{\mathrm{cyc}}(\mathcal{C}, -) : \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Sp}$ and $\mathrm{THH}^{\mathrm{cyc}}(\mathcal{C}, -) : \mathrm{Bimod}(\mathcal{C}) \to \mathrm{CycSp}$ converge to the limit of their Goodwillie-Taylor tower for $M$ 1-connective.

## 11.2 The Main Result

Assembling Corollary 11.3 and Corollary 8.7, we get the following result:

**Theorem 11.4** Let $(\mathcal{C}, M)$ be a bounded heart category and $M$ such that $\Omega M$ is a weighted bimodule. Then, there is a cartesian square of spectra:

$$\begin{tikzcd} \mathrm{K}^{\mathrm{lace}}(\mathcal{C}, M) \arrow[r] \arrow[d] & \mathrm{TC}^{\mathrm{lace}}(\mathcal{C}, M) \arrow[d] \\ \mathrm{K}(\mathcal{C}) \arrow[r] & \mathrm{TC}(\mathcal{C}) \end{tikzcd}$$

and the common fiber of the vertical maps is given by $\mathrm{TR}(\mathcal{C}, M)$.

We note that in the current version of this text, this result depends of Theorems 7.18, 7.25, 7.28 and 8.4 for which we did not give a proof but only refered to [HNS25].

This theorem has a famed precedent, proven by Dundas-Goodwillie-McCarthy. In [DGM13], they prove a different statement which applies to general nilpotent extensions of connective ring spectra but specializes in the split square-zero case to the Corollary below. Let us explain how to deduce this split square-zero version it from ours; we will not explain the nilpotent case as it is not a straightforward consequence of the above theorem and requires more work.

Recall that if $R$ is connective and $M$ is a connective $R$-module, then Example 3.10 shows:

$$\mathrm{Lace}(\mathrm{Perf}(R), \Sigma M \otimes_R -) \simeq \mathrm{Perf}(R \oplus M)$$

Moreover, $\mathrm{Perf}(R)$ does indeed carry a heart structure, in fact it is a weight structure, whose heart is $\mathrm{Proj}(R)$, and thanks to the shift, weighted bimodules $M$ exactly correspond to $(-1)$-connective $R$-bimodules. In particular, we get

**Corollary 11.5 — Dundas-Goodwillie-McCarthy.** Let $R$ be a connective ring spectrum and $M$ a connective $R$-bimodule. There is a cartesian square of spectra:

$$\begin{tikzcd} \mathrm{K}(R \oplus M) \arrow[r] \arrow[d] & \mathrm{TC}(R \oplus M) \arrow[d] \\ \mathrm{K}(R) \arrow[r] & \mathrm{TC}(R) \end{tikzcd}$$

where $R \oplus M$ denotes the (split) square-zero extension of $R$ by $M$.

Remark that the above discussion with $\mathcal{C}$ having a weight structure, but not necessarily of the form $\mathrm{Perf}(R)$, also recovers the split case of the main theorem of [ES21].

## 11.3 Examples of heart categories

### 11.3.1 Stacks with the derived resolution property

To fill the above discussion with more examples of heart categories $\mathcal{C}$, whose heart $\mathcal{E}$ is sufficiently small that it admits functors $\mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathrm{Sp}_{\geq 0}$. Let us recall a result of [Sau23b].

For a quasi-compact quasi-separated scheme $X$, we denote $\mathrm{Perf}(X)$ the category of compact $\mathcal{O}_X$-modules.

We call a scheme $X$ *divisorial* if it has an ample family of line bundles as defined in [TT90, Definition 2.1] (see also [BGI71, Exposé II, 2.2.3]). For instance, quasi-projective varieties over a field are divisorial by Example 2.1.2 of [TT90]. We will denote $\mathrm{Vect}(X)$ the subcategory[14] of such complexes concentrated in degree 0.

[14] This is in fact a 1-category.

**Proposition 11.6 — Proposition 2.18 of [Sau23b].** Let $X$ be a divisorial scheme. Then, the category $\mathrm{Perf}(X)$ has a heart structure whose heart is $\mathrm{Vect}(X)$.

Thanks to Dhyan Aranha and Adeel Khan, we realized that the above statement is actually significantly more general, and applies to quasi-coherent quasi-separated derived Artin stack with the derived resolution property, we set out to prove it. Recall the following definition from [Kha21, Definition 1.32].

**Definition 11.7** Let $\mathcal{X}$ be a quasi-coherent quasi-separated derived Artin stack. We say that $\mathcal{X}$ has the *resolution property* if for every quasi-coherent module $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$, there exists a small family $\mathcal{E}_\alpha$ of locally free sheaves and a morphism

$$\bigoplus_\alpha \mathcal{E}_\alpha \twoheadrightarrow \mathcal{F}$$

which is surjective on $\pi_0$.

We say that $\mathcal{X}$ has the *derived resolution property* if this holds more generally for quasi-coherent complexes $\mathcal{F}$.

The derived property is stronger than the non-derived. Remark that if $\pi_0(\mathcal{F})$ is supposed to be of finite type, then one can equivalently ask for a family $(\mathcal{E}_\alpha)$ composed of exactly one finite locally free sheaf.

**Definition 11.8 — Aranya-Khan-Ravi.** Let $\mathcal{C}$ be a stable category. A *resolving structure* on $\mathcal{C}$ is the datum of a sequence

$$\mathcal{C}_0 = \mathcal{C}_{\leq 0} \subset \mathcal{C}_{\leq 1} \subset \ldots \subset \mathcal{C}_{\leq \infty}$$

of subcategories of $\mathcal{C}$ such that

(i) The full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ is closed under finite direct sums; in particular, it is additive.

The full subcategory $\mathcal{C}_{\leq\infty} \subset \mathcal{C}$ is closed under finite colimits and extensions; in particular, it is prestable (see [Lur18, Cor. C.1.2.3]).

(ii) For every object $X \in \mathcal{C}$, there exists an integer $i \geq 0$ such that $\Sigma^i X \in \mathcal{C}_{\leq\infty}$

(iii) Suppose given an exact sequence $X \to Y \to Z$ in $\mathcal{C}$. If $X, Y \in \mathcal{C}_{\leq n}$ for some $n \geq 0$, then $Z \in \mathcal{C}_{\leq n+1}$. If $Y, Z \in \mathcal{C}_{\leq n}$, for some $n \geq 0$, then $X \in \mathcal{C}_{\leq n}$ .

(iv) For every object $X \in \mathcal{C}_{\leq n}$, where $n \geq 0$, there exists an exact sequence

$$H \to X \to Y$$

where $H \in \mathcal{C}_0$ and $D \in \mathcal{C}_{\leq n}$.

**Lemma 11.9** Let $\mathcal{C}$ be a stable category and suppose that $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ are two full subcategories such that $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ are closed under extensions, $\mathcal{C}_{\geq 0}$ under finite colimits and $\mathcal{C}_{\leq 0}$ under finite limits, and further, every object lies in $\mathcal{C}_{\geq -m}$ for some $m \geq 0$.

Then the following are equivalent:

(1) For every $X \in \mathcal{C}$, there exists an exact sequence $Y \to X \to Z$ such that $Y \in \mathcal{C}_{\leq 0}$ and $Z \in \mathcal{C}_{\leq 1}$. In particular, the pair of categories determines a heart structure on $\mathcal{C}$.

(2) For every $X \in \mathcal{C}_{\geq 0}$, there exists an exact sequence $H \to X \to Z$ such that $H \in \mathcal{C}_{[0,0]} := \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0}$ and $Z \in \mathcal{C}_{\geq 1}$.

**Proof.** (1) $\implies$ (2) follows from Lemma 10.9.

By induction, (2) $\implies$ (1) follows if, assuming (2), we can provide weight decompositions for every $X \in \mathcal{C}_{\geq -1} = \Omega\mathcal{C}_{\geq 0}$. Using the hypothesis for $\Sigma X$, we have a sequence:

$$Y \longrightarrow \Sigma X \longrightarrow Z$$

with $Y \in \mathcal{C}_{[0,0]}$ and $Z \in \mathcal{C}_{\geq 1}$. We can apply (2) again to $\Omega Z$, giving a sequence

$$Y' \longrightarrow \Omega Z \longrightarrow Z'$$

again with the according hypotheses on $Y' \in \mathcal{C}_{[0,0]}$ and $Z'$. We can thus form the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Omega Y \arrow[r, equal] \arrow[d] & \Omega Y \arrow[d] \\
\Omega Z' \arrow[r] \arrow[d, equal] & Y' \times_{\Omega Z} Y \arrow[r] \arrow[d] & X \arrow[d] \\
\Omega Z' \arrow[r] & Y' \arrow[r] & \Omega Z
\end{tikzcd}$$

where we looped the first sequence and rotated the second. In particular, by closure under extension of $\mathcal{C}_{\leq 0}$, we have that $Y' \times_{\Omega Z} Y \in \mathcal{C}_{\leq 0}$. It follows that

$$Y' \times_{\Omega Z} Y \longrightarrow X \longrightarrow Z'$$

is an exact sequence as wanted. □

The above lemma means that given a heart structure where every objects is bounded below in weight, the category $\mathcal{C}_{\leq 0}$ is unnecessary for the datum, as it can be recovered as the closure under extensions and limits of whatever terms appear on the left hand side of the decomposition

**Proposition 11.10** The datum of a resolving structure on $\mathcal{C}$ determines a heart structure on $\mathcal{C}$ which is bounded below. If further $\mathcal{C}_{\leq \infty} = \cup_n \mathcal{C}_{\leq n}$, then the structure is bounded.

**Proof.** We define $\mathcal{C}_{\geq 0} = \mathcal{C}_{\leq \infty}$ and $\mathcal{C}_{\leq 0}$ to be the closure under limits and extensions of $\mathcal{C}_0$ in $\mathcal{C}$. Remark that by (ii) of the definition, the extra condition of the above lemma is verified. In particular, to show that the structure is a heart structure, it suffices to provide decompositions for objects in non-negative weight, which we have by (iv).

To check that the structure is bounded, remark that every object is generally bounded below by (ii). Assuming the extra condition, every $X \in \mathcal{C}_{\leq \infty}$ becomes automatically bounded above as it can be obtained via finitely many resolutions with terms in the heart (whose delooping automatically land in $\mathcal{C}_{\leq 0}$. □

**Proposition 11.11** Suppose $\mathcal{C}$ is a bounded heart structure, then

$$\mathcal{C}^{\heartsuit} \subset \ldots \subset \mathcal{C}_{[0,n]} \subset \ldots \subset \mathcal{C}_{\geq 0}$$

defines a resolving structure on $\mathcal{C}$ such that $\mathcal{C}_{\geq 0} = \cup_n \mathcal{C}_{[0,n]}$.

**Proof.** Condition (i) is straightforward from the definition. Condition (ii) follows from the boundedness. Condition (iii) is subsumed by Lemma 10.9. Condition (iv) is given by the resolution of the definition. □

Given a quasi-coherent quasi-separated derived Artin stack $\mathcal{X}$, we write $\mathrm{Perf}(\mathcal{X})$ for the category of perfect complexes of $\mathcal{O}_{\mathcal{X}}$-modules and $\mathrm{Vect}(\mathcal{X})$ for the full subcategory spanned by those which are locally-free of finite type.

**Theorem 11.12** Let $\mathcal{X}$ be a quasi-coherent quasi-separated derived Artin stack with the derived resolution property. Then $\mathrm{Perf}(\mathcal{X})$ has a bounded heart structure with heart $\mathrm{Vect}(\mathcal{X})$.

**Proof.** By the above, to prove that $\mathrm{Perf}(\mathcal{X})$ carries a heart structure with heart $\mathrm{Vect}(\mathcal{X})$, it suffices to produce a resolving structure on $\mathrm{Perf}(\mathcal{X})$ with $\mathcal{C}_0 := \mathrm{Vect}(\mathcal{X})$. We let $\mathcal{C}_n$ denote the full subcategory of $\mathrm{Perf}(\mathcal{X})$ spanned by complexes of non-negative Tor-amplitude $\leq n$, and $\mathcal{C}_\infty$ their colimit, i.e. the full subcategory of complexes of finite non-negative Tor-amplitude. Properties (i) and (iii) holds without assumptions on $\mathcal{X}$, and similarly for property (ii) which follows from the fact that every perfect complex has bounded Tor-amplitude.

For property (iv), if $X \in \mathrm{Perf}(\mathcal{X})$ is of finite non-negative Tor-amplitude, then the resolution property implies that there is a morphism $Y \to X$ with $Y \in \mathrm{Vect}^{tf}(\mathcal{X})$ which is surjective on $\pi_0$, so in particular, the kernel has again finite non-negative Tor-amplitude, which concludes. $\square$

In particular, we deduce from the theorem of the heart for K-theory, see Theorem 10.18, the following statement:

**Corollary 11.13** Let $\mathcal{X}$ be a quasi-coherent quasi-separated derived Artin stack with the derived resolution property. Then, the map

$$\mathrm{K}(\mathrm{Vect}(\mathcal{X})) \xrightarrow{\simeq} \mathrm{K}(\mathrm{Perf}(\mathcal{X}))$$

is an equivalence.

### 11.3.2 Neeman's approximable categories

We adapt the definition of Neeman in [Nee21a] to the higher categorical world.

**Definition 11.14** Let $\mathcal{C}$ be a stable category which is generated by a single compact object (in particular presentable). We say that $\mathcal{C}$ is *approximable* if there is a compact generator $G \in \mathcal{C}^\omega$, a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and an integer $n \geq 0$ such that

(i) $\Sigma^n G \in \mathcal{C}_{\geq 0}$ i.e. $G$ is $(-n)$-connective and $\Sigma^{-n} G$ is (left)-orthogonal to $\mathcal{C}_{\geq 0}$ the connective objects, i.e. for every $X \in \mathcal{C}_{\geq 0}$,
$$\mathrm{map}_{\mathcal{C}}(\Sigma^{-n} G, X)$$
is 1-connective.

(ii) For every connective $F \in \mathcal{C}_{\geq 0}$, there is an exact sequence
$$E \longrightarrow F \longrightarrow D$$
with $D \in \mathcal{C}_{\geq 1}$ 1-connective and $E \in \overline{\langle G \rangle}_n^{[-n,n]}$, where $\overline{\langle G \rangle}_n^{[-n,n]}$ is the smallest subcategory of $\mathcal{C}$, containing $\Sigma^i G$ for $i$ ranging in the integer interval $[-n, n]$ and stable under direct sums, retracts and extensions.

A priori, this sounds like a terrible definition: there may be multiple ways to exhibit $\mathcal{C}$ as an approximable category, depending on the choice of t-structure and of compact generator, but Neeman shows that actually, it is surprisingly canonical. Relatedly, we have the following:

**Theorem 11.15** Let $\mathcal{C}$ be approximable. Then, the category $\mathcal{C}^\omega$ of compact objects admits a heart structure whose heart $H$ has the following property: there exists a $n \geq 0$ such that, for every $X, Y \in H$, $\mathrm{map}_{\mathcal{C}}(X, Y)$ is bounded below by $n$.

**Proof.** The heart structure is actually adjacent to the t-structure (and so, as in Bondarko, they determine one another). For the sake of readability, we let $\mathcal{C}$ be the category of compact objects and $\mathrm{Ind}(\mathcal{C})$ the approximable category. Denote $\mathcal{C}_{\geq 0}$ the category of compact objects which are connective in the t-structure on $\mathrm{Ind}(\mathcal{C})$.

For every $X \in \mathcal{C}$, there is $n \geq 0$ such that $\Sigma^n X \in \mathcal{C}_{\geq 0}$. This holds for $G$ by assumption and so for every object in the smallest idempotent-complete category generated by $G$ in $\mathcal{C}$. But $G$ is generating so this is all of $G$.

In particular, we are in the situation of Lemma 10.9 so it suffices to provide resolutions for objects in $\mathcal{C}_{\geq 0}$. Given (ii), this is clearly the case hence we find that we can choose the heart structure to be such that $\mathcal{C}^\heartsuit$ is $\overline{\langle G \rangle}_n^{[-n,n]}$ and $\mathcal{C}_{\leq 0}$ the closure under finite limits and extensions of the above.

To conclude, it suffices to check that mapping spectra in $\overline{\langle G \rangle}_n^{[-n,n]}$ are bounded below. But we also have that
$$\mathrm{map}(\Sigma^{-n} G, X)$$
is 1-connective for every object in $\mathcal{C}_{\geq 0}$, so in particular in the heart. Given the definition of

$\overline{\langle G\rangle}_n^{[-n,n]}$, this implies that for every $H, H' \in \mathcal{C}^\heartsuit$

$$\mathrm{map}(H, H')$$

is $1-(2n+1) = -2n$ connective (where the $2n+1$ comes from the size of $[-n,n]$). □

**Remark 11.16** In particular, such hearts are quite far from being stable: they live in the right-orthogonal complement of $\mathbf{Cat}^{\mathrm{Ex}}$ within $\mathbf{Exact}_\infty$. This means that there are no non-zero exact functor from a stable category to them.

It is a theorem of Bondal-van den Bergh in [BvdB03] that if $X$ is a qcqs scheme then $D_{qc}(X)$ is compactly-generated by one object so that it is legitimate to wonder whether $D_{qc}(X)$ is compactly generated. If $X$ is actually separated, then Neeman has shown that $D_{qc}(X)$ is in fact approximable. If $X$ is separated of finite type over a noetherian ring, this is [Nee21b, Theorem 5.8] and the generalization is [Nee21c, Example 3.6]. In consequence:

**Corollary 11.17** If $X$ is a quasi-compact separated scheme, then there is a bounded heart structure on the stable category $\mathrm{Perf}(X)$ such that the mapping spectra of $\mathrm{Perf}(X)^\heartsuit$ are uniformly bounded below, i.e. there is an integer $n \geq 0$ such that:

$$\mathrm{map}_{\mathrm{Perf}(X)}(X, Y) \text{ is } (-n) \text{ connective for every } X, Y \in \mathrm{Perf}(X)^\heartsuit$$

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