The "fundamental" theorem of localizing invariants

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What's algebraic K-theory

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Many questions about R can be turned into questions about $K(R)$.

What's algebraic K-theory (2)

Two examples of problems which can be formulated via K-theory:

- (Wall's finiteness obstruction) Let X be a finitely dominated space, i.e. there is a finite CW-complex Y which retracts onto X. Is X a finite CW-complex itself? The obstruction lies in $\tilde{K}_0\mathbb{Z}[\pi_1(X)]$.

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but also the (solved) Quillen-Lichtenbaum conjecture, and many others.

Algebraic K-theory for (higher) categories

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Claim

Algebraic K-theory of stable ∞-category is a sweet spot.

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Theorem [Bass-Heller-Swan]

There is an isomorphism

$$K_0(R[t, t^{-1}]) \simeq K_0(R) \oplus K_{-1}(R) \oplus NK_0^{+}(R) \oplus NK_0^{-}(R)$$

where the $NK_0^{\pm}(R)$ are isomorphic to one another[a], vanishing for normal rings.

[a] and to the $NK_0(R)$ above

Negative K-groups

A new group has appeared: $K_{-1}(R)$. This group is by definition the cokernel of the map $K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t, t^{-1}])$, but it does not appear in our definition for algebraic K-theory: $K(R)$ is a *connective* spectrum.

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Non-connective K-theory

There is a Morita-invariant functor $\mathbb{K}$ taking a ring R to a (generally non-connective) spectrum $\mathbb{K}(R)$, such that:

- For $n \geq 0$, there are isomorphism $K_n(R) \simeq \mathbb{K}_n(R)$, i.e. $K(R)$ is the connective cover of $\mathbb{K}(R)$.
- $\pi_{-1}\mathbb{K}(R) \simeq K_{-1}(R)$ as defined above.

The "fundamental" theorem of algebraic K-theory

With this new non-connective K-theory functor, there is a neat Bass-Heller-Swan formula for the entire spectrum:

Fundamental Theorem for non-connective K-theory

We have an equivalence of spectra

$$\mathbb{K}(R[t, t^{-1}]) \simeq \mathbb{K}(R) \oplus \Sigma\mathbb{K}(R) \oplus N\mathbb{K}_{+}(R) \oplus N\mathbb{K}_{-}(R)$$

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Passing to connective covers, one recovers a formula for regular K-theory with an extra group in π_0, coming from the connective cover of $\Sigma\mathbb{K}(R)$. This is the *canonical non-connective delooping of K-theory* that one finds in the classical statements of Quillen/Grayson.

"Fundamental" theorems ?

Questions

- Can we have the same for the *sweet spot*, i.e. stable ∞-categories.
- Can we simplify parts of the proof there ?
- Can we generalize the formula to other invariants related to K-theory, say THH, TC, KH, etc ... ?

To do this, we have to talk in more details about the properties of algebraic K-theory of stable ∞-categories.

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Stable ∞-categories

Let $\mathcal{C}$ be an ∞-category.

Definition

$\mathcal{C}$ is said to be *stable* if the following are satisfied:

- $\mathcal{C}$ is pointed, i.e. has a zero object
- $\mathcal{C}$ admits finite limits and finite colimits.
- Given a square in $\mathcal{C}$:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & T \end{array}$$

  the square is cocartesian if and only if it is cartesian.

This is a property of an ∞-category, not a structure.

Examples of stable ∞-categories

Examples (Motivating)

The ∞-category Sp of spectra, whose homotopy category is the stable homotopy category, is stable (hence the name).
However, the ∞-category of spaces is not stable. It can be stabilized and this yields the above category of spectra.

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Examples

For every simplicial set K and every stable $\mathcal{C}$, the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ of functors from K to $\mathcal{C}$ is also stable.

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Examples

If $\mathcal{A}$ is abelian, there is a stable ∞-category $D(\mathcal{A})$ whose homotopy category is the ordinary *derived category of* $\mathcal{A}$.

Exact functors

Definition

Let $\mathcal{C}$, $\mathcal{D}$ be stable ∞-categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be exact if it preserves finite limits or finite colimits, in which case it preserves both.

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Denote Cat_∞^{Ex} the (non-full!) subcategory of Cat_∞ spanned by stable ∞-categories and exact functors.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

Definition

F is a *localisation* at some class $\mathcal{W}$ of arrows in $\mathcal{C}$ if for every $\mathcal{E}$, precomposition by F induces an equivalence

$$F^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{W}}(\mathcal{C}, \mathcal{E})$$

F is a *left Bousfield localisation* if it has a fully-faithful right adjoint. This condition implies the equivalence above.

Localization of stable ∞-categories

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For exact functors between stable ∞-categories, localizations are *Verdier quotients*.

Verdier sequences

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If $\mathcal{C}$ is furthermore closed under retracts in $\mathcal{D}$, then the above is also a fiber sequence. Such sequences are called *Verdier sequences*.

Claim

Every localization between stable ∞-categories is the localization at a Verdier quotient. Every fiber-cofiber sequence is a Verdier sequence.

Verdier-localizing invariants

Let $\mathcal{E}$ be a presentable stable ∞-category (which will be Sp most of the time).

Definition

A functor $F : \mathrm{Cat}_{\infty}^{Ex} \to \mathcal{E}$ is a Verdier-localizing invariant if it sends Verdier sequences to fiber sequences.

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Algebraic K-theory $K : \mathrm{Cat}_\infty^{Ex} \to \mathrm{Sp}$ is Verdier-localizing.
Non-connective K-theory $\mathbb{K}$ is also Verdier-localizing.

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Examples

Topological Hochschild homology THH is Verdier-localizing and so is TC, topological cyclic homology.

Karoubi equivalences

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Definition

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If $f : \mathcal{C} \to \mathcal{D}$ is localization of stable ∞-categories, then $\mathrm{Idem}(f)$ is almost a localization: it is a localization on its essential image, and the inclusion of the essential image is a Karoubi-equivalence.

Karoubi-localizing invariants

Definition

A functor $F : \mathrm{Cat}_{\infty}^{Ex} \to \mathcal{E}$ is Karoubi-localizing if it is Verdier-localizing and inverts Karoubi equivalences.

Examples

$\mathbb{K}$, KH, THH and TC are Karoubi-localizing. However, K is *not* Karoubi-localizing. Thomason's cofinality theorem guarantees that $K_0(\mathcal{C}) \to K_0(\mathrm{Idem}(\mathcal{C}))$ is injective but there are instances where it is not surjective (for instance for $\mathcal{C} = \mathrm{Sp}^f$, the ∞-category of finite spectra).

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Tensoring with S^1

If R is a ring, then one can consider the rings $R[t]$ and $R[t, t^{-1}]$ of respectively polynomials and Laurent polynomials. There are maps

$$R[t] \longrightarrow R[t, t^{-1}], \qquad R[t^{-1}] \longrightarrow R[t, t^{-1}]$$

given by the two possible inclusion of polynomials into Laurent polynomials which preserve constant polynomials. Note that they correspond to localizing at $\{t\}$ or $\{t^{-1}\}$.

Tensoring with S^1 (2)

In the higher setting of stable ∞-categories, there is an analogue. If $\mathcal{C}$ is stable, then so is $\mathrm{Fun}(S^1, \mathcal{C})$: this is the category of objects of $\mathcal{C}$ with an action of $\mathbb{Z}$.

Definition-Proposition

There exists $S^1 \otimes \mathcal{C}$ a stable ∞-category with a map $\mathcal{C} \to S^1 \otimes \mathcal{C}$ inducing an equivalence for every stable $\mathcal{D}$:

$$\mathrm{Fun}^{Ex}(S^1 \otimes \mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{Ex}(\mathcal{C}, \mathrm{Fun}(S^1, \mathcal{D}))$$

We have similar definitions replacing S^1 by $S^1_+ := B\mathbb{N}_+$ and $S^1_- := B\mathbb{N}_-$ (these are equivalent but they correspond to the two different identifications of $B\mathbb{N}$ in $B\mathbb{Z}$).

The Projective Line

For a stable ∞-category $\mathcal{C}$, there are two maps

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Proposition

T_+ and T_- are Verdier projections, i.e. localizations at some class of arrows.

We can consider the following pullback square of stable ∞-categories:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{C}) & \longrightarrow & S^1_+ \otimes \mathcal{C} \\ \downarrow & & \downarrow \\ S^1_- \otimes \mathcal{C} & \longrightarrow & S^1 \otimes \mathcal{C} \end{array}$$

We say that $\mathbb{P}(\mathcal{C})$ is the *Projective Line* of $\mathcal{C}$.

The Projective Line under a Verdier-localizing invariant

Theorem

If F is a Verdier-localizing invariant, then:

$$\begin{array}{ccc} F(\mathbb{P}(\mathcal{C})) & \longrightarrow & F(S^1_+ \otimes \mathcal{C}) \\ \downarrow & & \downarrow \\ F(S^1_- \otimes \mathcal{C}) & \longrightarrow & F(S^1 \otimes \mathcal{C}) \end{array}$$

is a cartesian square.

This is follows from the pasting lemma and the stability of Verdier projections under pullback.

The ∞-categorical Projective Bundle formula

Suppose $\mathcal{C}$ is stable, **idempotent complete**.

Theorem

For F a Verdier-localizing invariant, we have splittings

$$F(\mathbb{P}(\mathcal{C})) \simeq F(\mathcal{C}) \oplus F(\mathcal{C}) \tag{1}$$
$$F(S^1_\pm \otimes \mathcal{C}) \simeq F(\mathcal{C}) \oplus N_\pm F(\mathcal{C}) \tag{2}$$

There are versions of (1) for rings and even E_∞-rings (and $\mathbb{P}^1(R)$ is the projective line scheme), where it is usually called the *Projective Bundle formula*.

The Main Result

By combining the last two results, we get:

Theorem

For F a Verdier-localizing invariant and $\mathcal{C}$ stable, idempotent complete, we have a splitting

$$F(S^1 \otimes \mathcal{C}) \simeq F(\mathcal{C}) \oplus \Sigma F(\mathcal{C}) \oplus N_+ F(\mathcal{C}) \oplus N_- F(\mathcal{C})$$

Examples

One can consider Karoubi-localizing F such that $N_\pm F$ vanishes. These are the stable ∞-categorical version of $\mathbb{A}^1$-invariant functors. For those F, the formula simplifies to

$$F(S^1 \otimes \mathcal{C}) \simeq F(\mathcal{C}) \oplus \Sigma F(\mathcal{C})$$

Consequences

Issue

When $\mathcal{C}$ is $\mathrm{Perf}(R)$ the stable, idempotent-complete ∞-category of compact objects of R-Mod, then $S^1 \otimes \mathcal{C}$ is not quite $\mathrm{Perf}(R[t, t^{-1}])$.

However, it is true that $\mathrm{Idem}(S^1 \otimes \mathrm{Perf}(R)) \simeq \mathrm{Perf}(R[t, t^{-1}])$! So when our invariants are Karoubi-localizing, the formula of the previous slide computes the correct thing. Hence, the following formulas are correct

$$\mathbb{K}(R[t, t^{-1}]) \simeq \mathbb{K}(R) \oplus \Sigma\mathbb{K}(R) \oplus N_{+}\mathbb{K}(R) \oplus N_{-}\mathbb{K}(R)$$

$$THH(R[t, t^{-1}]) \simeq THH(R) \oplus \Sigma THH(R) \oplus N_{+} THH(R) \oplus N_{-} THH(R)$$

$$TC(R[t, t^{-1}]) \simeq TC(R) \oplus \Sigma TC(R) \oplus N_{+} TC(R) \oplus N_{-} TC(R)$$

Consequences (2)

Let X be a space and denote $\mathbb{A}(X)$ the non-connective K-theory of $\mathrm{Fun}(X, \mathrm{Sp})^c$, the subcategory of compact objects of $\mathrm{Fun}(X, \mathrm{Sp})$. Then,

$$\mathbb{A}(X \times S^1) \simeq \mathbb{A}(X) \oplus \Sigma\mathbb{A}(X) \oplus N_+\mathbb{A}(X) \oplus N_-\mathbb{A}(X)$$

If we pass to connective covers, we get a (known) formula for $A(X)$, Waldhausen's A-theory functor (the finitely-dominated version).

What's next ?

Question

For an (ordinary) additive category $\mathcal{A}$ equipped with a self-equivalence $\phi : \mathcal{A} \to \mathcal{A}$, Lück and Steimle have a formula to compute the twisted Laurent polynomials $\mathcal{A}_\phi[t, t^{-1}]$. Can it be upgraded to stable ∞-categories?

Question

A recent 9-author collaboration has developed hermitian K-theory, with Poincaré ∞-categories taking the place of stable ones. Is there a Bass-Heller-Swan formula in this context as well?

Question

What about other theorems of algebraic K-theory that are known in the case of rings or ring spectra (one major candidate would be Dundas-Goodwillie-McCarthy)?

Questions ?

Thank you for your attention!