Worksheet 3 - Homotopy II

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Exercise 1   Loops and Suspensions

Let $\mathcal{C}$ be a model category with a zero objects. For $X \in \mathcal{C}$, we denote ΣX the homotopy colimit of the following diagram $0 \longleftarrow X \longrightarrow 0$ and ΩX the homotopy limit of $0 \longrightarrow X \longleftarrow 0$.

1. Compute ΩX in sSet_*, Top_*, $\mathrm{Ch}(\mathbb{Z})$.
2. Compute ΣX in sSet_*, Top_*, $\mathrm{Ch}(\mathbb{Z})$.
3. Show that $\Sigma : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ is adjoint to $\Omega : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$. In which of the previous cases is this adjunction an equivalence?

Exercise 2   Right properness and cartesian squares

Let $\mathcal{C}$ be a right proper model category. Let $Z \twoheadrightarrow T$ be a fibration, T a fibrant object and suppose we have a cartesian square as follows:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \twoheadrightarrow & T \end{array}$$

1. Show that the above square is also homotopy cartesian.
2. Recall the model structure on Cat of Exercise 9 of the Worksheet 1.
   a) Show that this model structure is right proper. (Use Exercice 11.2 of the Worksheet 1)
   b) Deduce that the loop functor $\Omega : \mathrm{Cat} \to \mathrm{Cat}$ is (equivalent to) the constant functor $\emptyset$. (You can also work in Cat_*, the category of pointed categories and reduced functor[1], to avoid a set-theoretic headache and show that the loop is also 0).
   c) Deduce that the suspension Σ is also constant equal to $*$ everywhere.

Exercise 3   Segal's Γ-spaces

Let Γ^{op} be the category of finite sets and partially defined maps between them (we write Γ^{op} to follow the historical conventions, but following the more modern point of view, we will never use the category Γ in the following, and work only with its opposite).

1. Show that Γ^{op} is equivalent to the category of pointed finite sets. For the rest of the exercise, we adopt this point of view.
2. Let A be an abelian monoid. If S is a finite pointed set, define $F_A(S) = A^S$ (the set of pointed maps, where A is canonically pointed by 0) and if $f : S \to T$ is a pointed map, then $F_A(f)$ maps (a_s) to $(\sum_{f(\sigma)=t} a_\sigma)_t$.
   a) Show that F_A is a functor $\Gamma^{op} \to \mathrm{Set}$ which sends coproducts to products and $*$ to the point.
   b) Show that the functor $\mathrm{AbMon} \to \mathrm{Fun}(\Gamma^{op}, \mathrm{Set})$ which maps A to F_A is fully-faithful.
   c) Reciprocally, let $F : \Gamma^{op} \to \mathrm{Set}$ be a functor sending disjoint unions to products and $\emptyset$ to the point. Show that $F(*)$ is an abelian monoid A such that $F \simeq F_A$.

[1] A pointed category is a category with a zero object, and a reduced functor sends zero to zero. Assume the model structure of Exercice 11.2 still applies.

A special Γ-space is a functor $F : \Gamma^{op} \to \mathrm{sSet}$ such that $F(*)$ is contractible and $F(X \coprod Y) \to F(X) \times F(Y)$ is a weak homotopy equivalence. More generally if $\mathcal{C}$ is a model category, a special Γ^{op}-object in $\mathcal{C}$ is a functor $F : \Gamma^{op} \to \mathcal{C}$ such that $F(*)$ is weakly equivalent to $*$ and $F(X \coprod Y) \to F(X) \times F(Y)$ is a weak homotopy equivalence.

3. (Segal condition) Let $F : \Gamma^{op} \to \mathcal{C}$ be a special Γ-object, show that there is a weak equivalence

$$F([n]) \longrightarrow \prod_{i=1}^{n} F([1])$$

4. a) Suppose $\mathcal{C}^{\otimes}$ is a special Γ-category. Show that $\mathcal{C}^{\otimes}([1])$ is endowed with the structure of a symmetric monoidal category.
   b) Reciprocally, if $(\mathcal{C}, \otimes)$ is a symmetric monoidal category, show that there exists a special Γ-category such that the evaluation at [1] endowed with the monoidal structure of the above question is monoidally-equivalent to $(\mathcal{C}, \otimes)$
   c) (Segal) Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category and denote $\iota\mathcal{C}$ its maximal subgroupoid. Show that the nerve $N(\iota\mathcal{C})$ acquires the structure of a special Γ-space.

Exercise 4 Quillen's Theorem A (*After Akhil Matthew*)

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We denote $N : \mathrm{Cat} \to \mathrm{sSet}$ the nerve functor.

1. a) Suppose there exists a natural transformation $\eta : F \implies G$ where $G : \mathcal{C} \to \mathcal{D}$ is another functor. Show that NF is homotopic to NG.
   b) Show that if F is a functor admitting an adjoint G, then NF is a homotopy equivalence and NG a homotopy inverse to NF.

Our goal is to show that if $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/}$ is contractible (i.e. its nerve is) for every $Y \in \mathcal{D}$, then NF is a homotopy equivalence. This fact is usually known as Quillen's Theorem A.

2. Show that there is a functor $\mathcal{D} \to \mathrm{Cat}$ who maps Y to the category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/}$.
3. a) Show that there is a map
$$\mathrm{colim}_{Y \in \mathcal{D}} N(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/}) \to N(\mathcal{C})$$
   b) Show that the n-simplices of $N(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/})$ are given by the data of a composable chain $X_0 \to \ldots \to X_n$ in $\mathcal{C}$ and a map $F(X_n) \to Y$ in $\mathcal{D}$. Deduce that the above map is a surjection in all degrees.
   c) Show that the above map is also injective in all degrees.
4. Suppose $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/}$ is contractible for every $Y \in \mathcal{D}$.
   a) Show that the maps $N(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/}) \to N(\mathcal{D} \times_{\mathcal{D}} \mathcal{D}_{Y/})$ induced by F are equivalences.
   b) Deduce that to have Quillen's Theorem A, it suffices to show that there is an isomorphism:
$$\mathrm{hocolim}_{Y \in \mathcal{D}} N(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/}) \simeq \mathrm{colim}_{Y \in \mathcal{D}} N(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/})$$
5. a) Recall the generating cofibrations in the projective model structure of $\mathrm{Fun}(\mathcal{D}, \mathrm{sSet})$.
   b) Denote F_i the (pointwise) i-skeleton of $Y \mapsto N(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/})$. Compute the n-simplices of the following pushout:
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(D, -) \times \partial\Delta^n & \longrightarrow & F_{n-1} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}}(D, -) \times \Delta^n & \longrightarrow & \mathcal{P} \end{array}$$
   Deduce that if F_{n-1} was cofibrant in the projective model structure of $\mathrm{Fun}(\mathcal{D}, \mathrm{sSet})$, then so is F_n.
   c) Deduce that the functor $Y \mapsto N(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{Y/})$ is cofibrant in the projective model structure of $\mathrm{Fun}(\mathcal{D}, \mathrm{sSet})$. Conclude.
6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor satisfying the above hypothesis. Show that there is a Quillen-equivalence of the projective model structures model $\mathrm{sSet}^{N\mathcal{D}} \simeq \mathrm{sSet}^{N\mathcal{C}}$ induced by precomposition by F. Deduce that:
$$\mathrm{hocolim}\, K \simeq \mathrm{hocolim}\, K \circ NF$$
   for every $K : N\mathcal{D} \to \mathrm{sSet}$.

Exercise 5 More on cofinality

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be cofinal if, for every $Y \in \mathcal{D}$, the category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/Y}$ is contractible. From the preceding exercise, we have seen that a cofinal functor $F : \mathcal{C} \to \mathcal{D}$ induces a homotopy equivalence $NF : \mathcal{C} \to \mathcal{D}$.

1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors and suppose F is cofinal. Show that G is cofinal if and only if $G \circ F$ is.
2. Show that the map $\Delta^{op} \to \Delta^{op} \times \Delta^{op}$ is cofinal. (Hint: it might be easier to work with $\Delta \to \Delta \times \Delta$ and to show the dual result).
3. Denote Δ_{inj} the subcategory of Δ where we only keep injective maps. Show that $\Delta^{op}_{inj} \to \Delta^{op}$ is cofinal.

Exercise 6 Towards ∞-categories

1. Show that the nerve $N : \mathrm{Cat} \to \mathrm{sSet}$ is a fully-faithful functor whose essential image is characterized by the following *unique* lifting property (also known as the Segal condition):

$$\begin{array}{ccc} \Delta^1 \times_{\Delta^0} \dots \times_{\Delta^0} \Delta^1 & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

   where $\Delta^1 \times_{\Delta^0} \dots \times_{\Delta^0} \Delta^1 \to \Delta^n$ is the *spine inclusion*, induced by the inclusions of successive arrows $[k] \to [k+1]$.

2. a) Show that the above lifiting property is equivalent to the following unique lifting properties for $1 \leq k \leq n-1$:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

   where Λ^n_k is the k^{th}-horn, obtained from Δ^n by erasing the interior and the k^{th} face.

   b) Deduce that the nerve of a category is a groupoid if and only if it is a Kan complex.