Worksheet 2 - Homotopy II

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Exercise 1 Quillen adjunctions

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be two left Quillen functors.

1. Show that $G \circ F$ is a left Quillen functor.
2. Show that there is a natural equivalence $\mathbb{L}(G \circ F) \simeq (\mathbb{L}G) \circ (\mathbb{L}F)$.
3. Let (L, R) be an adjoint pair. Show that L is left Quillen if and only if R is right Quillen.
4. Suppose the restriction of a functor F to cofibrant objects preserves trivial cofibrations, show that F is left derivable. (Hint: Ken Brown's lemma).

Exercise 2 Slice model structure II

Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{A}$. Let $f : X \to Y$ be a morphism. Recall that we defined in the last exercise sheet a model structure on every slice category $\mathcal{A}_{/X}$.

1. Show that the functor $f_! : \mathcal{A}_{/X} \to \mathcal{A}_{/Y}$ which postcomposes by f admits a right adjoint f^* and describe it.
2. Show that the pair $(f_!, f^*)$ is a Quillen pair of adjoints.
3. Suppose $\mathcal{A}$ is right proper, i.e. weak equivalences are stable under pullback by fibrations, and that $f \in \mathcal{W}$. Show that the pair $(f_!, f^*)$ is a Quillen equivalence.
4. (Rezk) Suppose that for every weak equivalence f, the pair $(f_!, f^*)$ is a Quillen equivalence. Show that $\mathcal{A}$ is right proper.

Exercise 3 The Universal property of sSet

Let $\mathcal{C}$ be a locally presentable category and $F : \Delta \to \mathcal{C}$ a cosimplicial object. Denote R_F the functor $\mathcal{C} \to \mathrm{sSet}$ given by $R_F(X) := \mathrm{Hom}_{\mathcal{C}}(F(\Delta^\bullet), X)$; we say that R_F is the simplicial representation of F.

1. Show that the formula for $R_F(X)$ defines a simplicial set.
2. Show that R_F preserves limits. Since both $\mathcal{C}$ and sSet are locally presentable, it admits a left adjoint L_F. Show that this left adjoint is in fact the left Kan extension of F along the Yoneda embedding $j : \Delta \to \mathrm{sSet}$. If you can, describe $L_F(X)$ as a coequalizer.
3. Let $L : \mathrm{sSet} \rightleftarrows \mathcal{C} : R$ be an adjoint pair. Denote $F : \Delta \to \mathrm{sSet}$ the cosimplicial object given by the composition $L \circ j$. Show that $R \simeq R_F$.
4. What is the underlying cosimplicial object of the nerve-geometric realization adjunction?
5. Let $X \in \mathrm{sSet}$, what is the underlying cosimplicial object of $\mathrm{Map}(X, -)$, the internal hom object ?

Exercise 4 On the Kan model structure of sSet

We let Δ^n (or $\Delta[n]$ in the course notes) be the standard n-simplex, $\partial\Delta^n$ its interior, S^n the quotient of Δ^n by its interior and Λ^n_k the k^{th}-horn.

1. We write $\mathcal{I}$ for the set of maps $\partial\Delta^n \to \Delta^n$.
   a) Show that a map of simplicial sets $f : X \to S$ is injective in all degrees if and only if it belongs to $LLP(RLP(\mathcal{I}))$, i.e. it is a cofibration.

b) Suppose X is a Kan complex, show that for any simplicial set S, $\mathrm{Map}(S, X)$ is also a Kan complex.

Let X be a simplicial set and $x \in X$. Recall that if X is a Kan complex, we have denoted $\pi_n(X, x)$ the quotient of $\mathrm{Map}(S^n, X)$ by the equivalence relation $\sim$ generated by $f \sim g$ if there is a map $\phi : \Delta^1 \to \mathrm{Map}(S^n, X)$ whose source and target are f and g.

2. Let $f : K \to L$ be a map of Kan complexes. Show that f is a weak equivalence if and only f induces an isomorphism on every homotopy group as defined above.
3. Let X be a topological space and denote $\mathrm{Sing}_\bullet X$ the simplicial set $\mathrm{Hom}(\Delta^\bullet, X)$. Show that $\mathrm{Sing}_\bullet X$ is a Kan complex and $\pi_n(X, x) \simeq \pi_n(\mathrm{Sing}_\bullet X, x)$.
4. Let X be a Kan complex. Show that $\pi_0(X) \simeq \pi_0(\mathrm{Sing}_\bullet |X|)$. Deduce inductively that $\pi_n(X, x) \simeq \pi_n(\mathrm{Sing}_\bullet |X|, x)$.
5. Conclude to show that the Kan model structure on sSet is Quillen-equivalent to the classical model structure on Top.

Exercise 5 The Dold-Kan correspondence

1. Let $\mathcal{A}$ be an abelian category and $X_\bullet : \Delta^{op} \to \mathcal{A}$ a simplicial $\mathcal{A}$-object.
   a) Let $d^n := \sum_{i=0}^n (-1)^i d_i^n : X_n \to X_{n-1}$. Show that $d^n \circ d^{n+1} = 0$. (Hint: recall that $d_i^n \circ d_j^{n+1} = d_{j-1}^n \circ d_i^{n+1}$ when $i < j$).
   b) Show that d^n sends degenerate simplicies to 0. Deduce that there is a functor $N_* : \mathrm{Fun}(\Delta^{op}, \mathcal{A}) \to \mathrm{Ch}_{\geq 0}(A)$ obtained by applying the previous construction and then modding out the degenerate simplices.
2. Denote $\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Ab}$ the free abelian group functor. When we write $N_* : \mathrm{sSet} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$, we mean the precomposition of the above defined N_* for $\mathcal{A} = \mathrm{Ab}$ by $\mathbb{Z}[-]$.
   a) Using ideas of Exercice 3, show that if N_* is a left adjoint, then its right adjoint $\tilde{\Gamma}$ is necessarily given on objects by
   $$\bigoplus_{[n] \twoheadrightarrow [k]} A_k$$
   where the sum is taken over all surjections $[n] \to [k]$ of Δ.
   b) Show that N_* indeed admits a right adjoint, given by the above formula. Deduce that the composite:
   $$\mathrm{Fun}(\Delta^{op}, \mathrm{Set}) \xrightarrow{\mathbb{Z}[-]_*} \mathrm{Fun}(\Delta^{op}, \mathrm{Ab}) \xrightarrow{N_*} \mathrm{Ch}_{\geq 0}(\mathbb{Z})$$
   admits a right adjoint Γ.
3. (HA 1.2.3.11) Show that $N_n(\Gamma(A_\bullet))$ can be identified with the summand A_n of $\Gamma(A_\bullet)_n$. Deduce that there is a natural isomorphism of chain complex $\mathrm{id} \simeq N_*\Gamma$.
4. (HA 1.2.3.13) Let $\theta(X) : \Gamma(N_*(X)) \to X$ be the counit at X.
   a) Show that θ is injective on each degree.
   b) Show that θ is surjective on each degree.

We have just shown that Γ and N_* are inverses to one another. This is usually called the *Dold-Kan correspondence* in Ab.

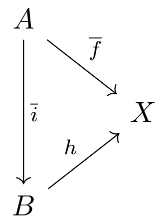
5. Suppose $\mathcal{A}$ is an idempotent-complete[1] abelian category. Using that the Yoneda embedding $j : \mathcal{A} \to \mathcal{A}' := \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Ab})$ is fully-faithful, deduce the Dold-Kan correspondence in $\mathcal{A}'$ and then in $\mathcal{A}$ itself.

Exercise 6 A Lifting Criterion (*HTT A.2.3*)

Let $\mathcal{C}$ be a model category, and denote $\bar{\cdot} : \mathcal{C} \to h\mathcal{C}$ the localisation functor. Suppose we have $i : A \to B$ a cofibration between cofibrant objects, and $f : A \to X$ a map with X fibrant. Suppose there is a commutative

[1] Here, you can take it to mean "closed under taking direct summand"

diagram in $h\mathcal{C}$



Show that there exists $g : B \to X$ such that $f = gi$ (in particular, we have $\overline{g} = h$).

Exercise 7   Monoidal model structures (*HTT A.3.1*)

Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ be three model categories and $F : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$ a functor. F is a *left Quillen bifunctor* if it preserves small colimits in each variable and for every cofibration $i : M \to M'$ in $\mathcal{M}$ and $j : N \to N'$ in $\mathcal{N}$, the induced map

$$i \wedge j : F(M, N') \coprod_{F(M,N)} F(M', N) \longrightarrow F(M', N')$$

is a cofibration, which is trivial as soon as either i or j is.

A monoidal model category is a closed[2] monoidal category $(\mathcal{S}, \otimes)$ equipped with a model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ with the following compatibility axioms:

- $\otimes$ is a left Quillen bifunctor
- The unit $\mathbb{1}$ is cofibrant

1. Show that the cartesian product of simplicial sets equipped with the Kan model structure is a left Quillen bifunctor. Deduce that the Kan model structure endows sSet with a monoidal model structure.
2. Let $(\mathcal{A}, \otimes)$ be a monoidal model category. Let $X \in \mathcal{A}$, show that $- \otimes X$ is a left Quillen functor.
3. Let $(\mathcal{A}, \otimes)$ be a monoidal model category.
   a) Show that the left derived tensor product $\otimes^L$ exists. (Hint: use Question 1.4)
   b) Show that $(h\mathcal{A}, \otimes^L)$ is a monoidal category.
   c) Show that the localization functor $\mathcal{A} \to h\mathcal{A}$ acquires a lax-monoidal structure which is strong monoidal on the restriction to cofibrant objects.
4. Show that the category of chain complexes with the usual tensor product:

$$(C_\bullet \otimes D_\bullet)_n := \bigoplus_{i+j=n} C_i \otimes D_j$$

   is a monoidal model category when equipped with the projective model structure.

(Schwede-Shippley) A monoidal model category $\mathcal{A}$ satisfies the *monoid axiom* if, for every acyclic cofibration j, the class of arrows generated by cobase change and transfinite composition by the $j \wedge \mathrm{id}_X$ for every $X \in \mathcal{A}$ is contained in $\mathcal{W}$, the class of weak equivalences.

5. Let $\mathcal{A}$ be a monoidal model category where every object is cofibrant. Show that $\mathcal{A}$ satisfies the monoid axiom.
6. Suppose $\mathcal{A}$ is cofibrantly generated, with $\mathcal{J}$ a set of generating acyclic cofibrations. Show that if for every $j \in \mathcal{J}$, the monoid axiom holds for j, then $\mathcal{A}$ satisfies the monoid axiom for every acyclic cofibration.

Exercise 8   Enriched model categories

[2]I.e. the one variable tensor $- \otimes X$ has a right adjoint $\underline{\mathrm{Hom}}(X, -)$

Recall that a $\mathcal{V}$-enriched category $\mathcal{C}$ is tensored and cotensored over a closed monoidal $\mathcal{V}$, if there are two functors $\otimes : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$ and $[-,-] : \mathcal{V}^{op} \times \mathcal{C} \to \mathcal{V}$ such that there are natural equivalences:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X_1 \otimes V, X_2) \simeq \underline{\mathrm{Hom}}_{\mathcal{V}}(V, \underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, X_2))$$
$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, [V, X_2]) \simeq \underline{\mathrm{Hom}}_{\mathcal{V}}(V, \underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, X_2))$$

of objects in $\mathcal{V}$.

1. Let $V \in \mathcal{V}$, show that $[V, -]$ is right adjoint to $V \otimes -$.
   Let $\mathcal{V}$ be a monoidal model category (see above). Let $\mathcal{A}$ be a $\mathcal{V}$-enriched category, cotensored and tensored over $\mathcal{V}$, and equipped with a model structure.
2. (HTT A.3.1.6) Show that the following propositions are equivalent:
   1. $\otimes : \mathcal{C} \otimes \mathcal{V} \to \mathcal{C}$ is a left Quillen bifunctor
   2. For any cofibration $i : D \to D'$ and any fibration $j : X \to Y$ in $\mathcal{A}$, the induced

      $$h : \underline{\mathrm{Hom}}_{\mathcal{C}}(C', X) \Longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(C, X) \times_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,Y)} \mathrm{Hom}(C', Y)$$

      is a fibration in $\mathcal{V}$, trivial as soon as i or j is.
   3. For any cofibration $i : V \to V'$ in $\mathcal{V}$ and any fibration $j : X \to Y$ in $\mathcal{A}$, the induced

      $$k : [V', X] \longrightarrow \ [V, X] \times_{[V,Y]} [V', Y]$$

      is a fibration in $\mathcal{A}$, trivial as soon as i or j is.

   A $\mathcal{A}$ satisfying the above is called a $\mathcal{V}$-enriched model category. In particular, every monoidal model category is enriched over itself.
3. Show that $h\mathcal{A}$ inherits a $\mathcal{V}$-enriched structure, such that $\mathcal{A} \to h\mathcal{A}$ is a $h\mathcal{V}$-enriched functor, and where the mapping objects are given by

   $$\underline{\mathrm{Hom}}_{h\mathcal{A}}(X, Y) \simeq \overline{\underline{\mathrm{Hom}}_{\mathcal{A}}(X, Y)}$$